Graphs (UNIT-IV)

Basic concepts

Graph theory was born in 1736, with Euler's paper on Konigsberg bridge problem. Euler the father of graph theory, solved this long pending problem by means of graph.

The term graph is used frequently to denote the diagram of the real valued function $y=f(x)$.

Definition:

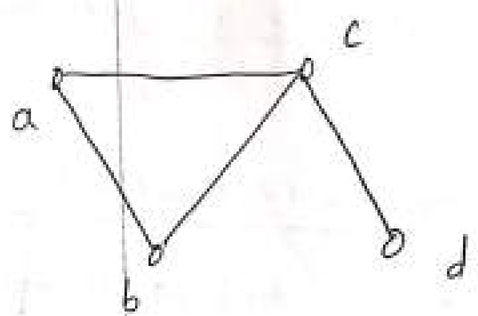
A Graph (denoted as $(G=(V,E))$ consist of a non-empty set of vertices or nodes 'V' and a set of edges 'E'.

Let us consider, A Graph is $G=(V,E)$.

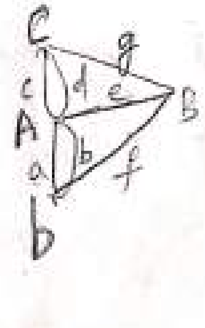
where $V=\{a,b,c,d\}$.

$E=\{\{a,b\},\{a,c\}\{b,c\}\{c,d\}\}$.

Graph



$G=(V,E)$.



Applications of graph:

Graphs are nothing but connected nodes (vertex). So, any network related, routing, finding relation, path etc.

Real Life Applications:

1. connecting with friends on social media, where each ~~vertex~~ user is a vertex, and when users connect they create an edge.
2. Using GPS (global positioning system) / google maps / yahoo maps to find a route based on shortest route.
3. Google, to search webpages, where pages on the internet are linked to each other by hyperlinks. each page is a vertex and the link between two pages is an edge.

* 4. Implementing newsfeed in facebook – Consider u as a root and all your adjacent vertex as your friends for each friend their adjacent vertex are their friends.

5. Implementing execution of object files – Dependency of object files to be which needs to be executed. (DFS in Graph).

Kinda

Even and odd vertex:-

If the degree of a vertex is even, the vertex is called an even vertex.

and if the degree of a vertex is odd, the vertex is called an odd vertex.

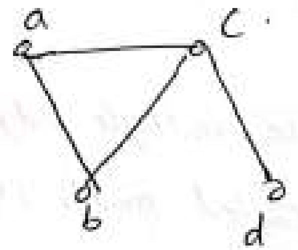
Degree of vertex:

The degree of vertex 'V' of graph G (denoted by (deg(v)) is the number of edges incident or adjacency with the vertex 'V'.

Degree of graph:-

The degree of a graph is the largest vertex degree of that graph.

Example:



vertex	degree	Even/odd
a	2	Even
b	2	Even
c	3	odd
d	1	odd.

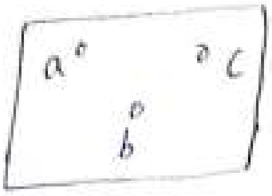
Note:- ~~For the~~ Degree of the above graph is 3

Basic definitions

① Null graph:

A Null graph has no edges. The null graph of n vertices is denoted by N_n.

Ex:



② Simple graph:

A graph is called simple graph / strict graph if the graph is undirected and does not contain any loop or multiple edges.

Ex:

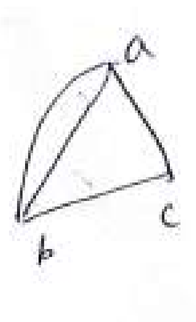
③ Multi graph:

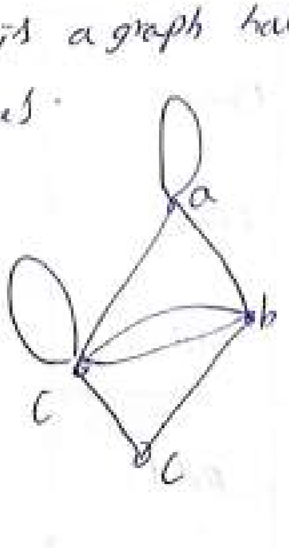
A Graph contains the multiple edges between the same set of vertices, it is called multi graph.

(or).

In other words, it is a graph having atleast one loop or multiple edges.

Ex:

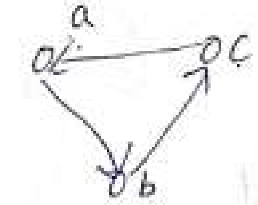




④ Directed graph:

A graph $G=(V,E)$ is called a directed graph if the edge set is made of ordered vertex pair.

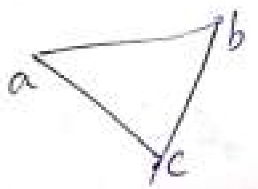
Ex:



⑤ undirected graph

A graph is called if the edge set is made of unordered vertex pair.

Ex:

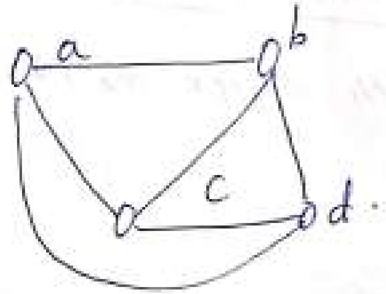


⑥ Regular graph:

A graph is regular if all the vertices of the graph have the same degree.

In a regular graph G of degree r, the degree of each vertex of G is r.

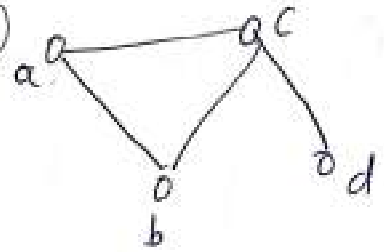
Ex:

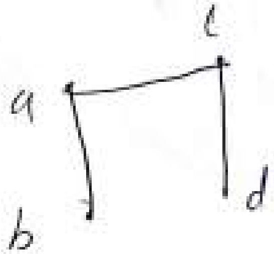


⑦ Complete graph:

(7) Connected graph

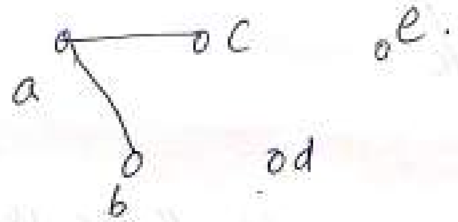
A graph is connected if any two vertices of the graph are connected by a path.

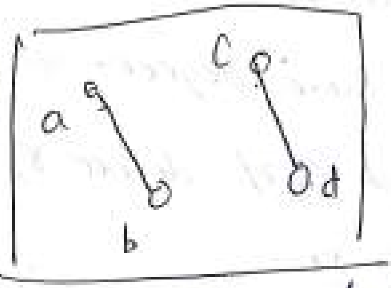
Ex: (1) 

(2) 

(8) Disconnected graph:

A graph is disconnected if atleast two vertices of the graph are not connected by a path.

Ex: (1) 

(2) 

(9) ~~Degree of vert~~ Pendent vertex:

A vertex with degree one is called pendent vertex.

Ex: 

(10) Isolated vertex:

A vertex with degree zero is called an isolated vertex. Ex: 

(11) parallel Edges:

In a graph if a pair of vertices is connected by more than one edge, then those edges are called parallel edges.

Ex:

In the above graph a and b are the two vertices which are connected by two edges ab and ab between them. So it is called as a parallel edge.

(12) Trivial graph

A graph with only one vertex is called a trivial graph.

Ex: • a

Here the only one vertex 'a' with no other edges. Hence it is a trivial graph.

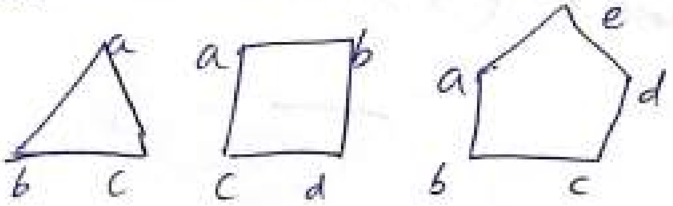
(13) cycle graph

A simple graph with 'n' vertices ($n \geq 3$) and 'n' edges is called a cycle graph if all its edges form a cycle of length 'n'.

(or)

If the degree of each vertex in graph is two. then it is called a cycle graph (C_n).

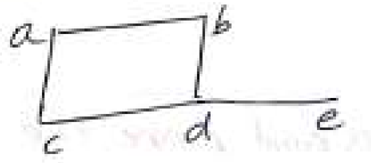
Ex:



Degree Sequence of a graph

If the degree of all vertices in a graph are arranged in descending or ascending order then the sequence obtained by is known as the degree sequence of graph.

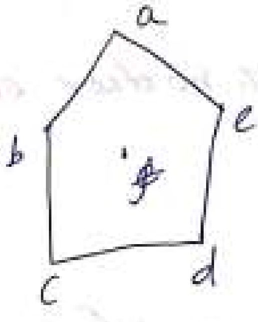
Ex:



vertex:	a	b	c	d	e
connected to	b,c	a,d	a,d	c,b,e	d
degree	2	2	2	3	1

In the above graph, for the vertices {d, a, b, c, e}, the degree sequence {3, 2, 2, 2, 1}.

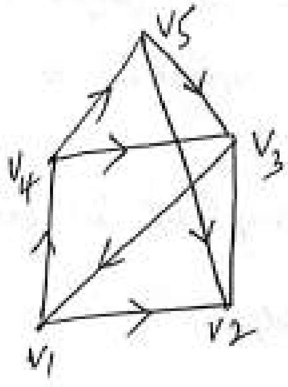
Ex:



vertex:	a	b	c	d	e	f
connected to:	b,e	a,c	b,d	c,e	a,d	
degree:-	2	2	2	2	2	0

In the above graph, for the vertices {a, b, c, d, e, f}, the degree sequence is {2, 2, 2, 2, 2, 0}.

Find the degree of the following ~~gr~~ directed graph.



indeg(v_1) = 1   outdeg(v_1) = 2
indeg(v_2) = 2   outdeg(v_2) = 1
indeg(v_3) = 3   outdeg(v_3) = 1
indeg(v_4) = 1   outdeg(v_4) = 2
indeg(v_5) = 1   outdeg(v_5) = 2.

Total deg (v_1) = 3
total deg (v_2) = 3.
total deg (v_3) = 4
total deg (v_4) = 3
total deg (v_5) = 3.

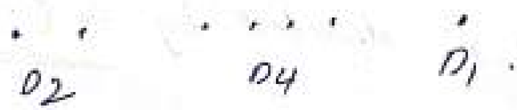
Types of graphs

There are five types of graphs.

1. Discrete graph
2. complete graph (full graph)
3. Linear graph
4. Bipartite graph (partition graph)
5. complete Bipartite graph.

1. Discrete graph:

For each integer $n \geq 1$, D_n denotes the discrete graph with 'n' vertices and no edges. It is order of 'n'.

Ex: 

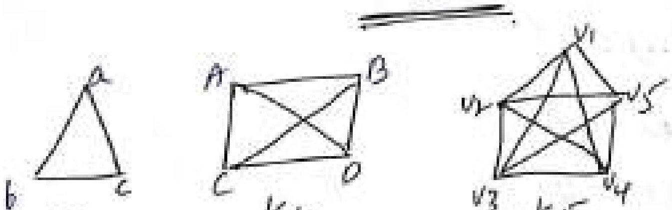
(2) Complete graph (full graph)

For each integer, K_n denotes the graph with 'n' vertices and each vertex is adjacent to every other vertex.

K_n is called the complete graph of 'n' vertices.

In K_n there will be a $^nC_2$ edges.

i.e $\frac{n(n-1)}{2}$ edges.

Ex:-  are complete graphs

K_3 K_4 K_5

(3) Linear graph:

For each integer $n \geq 1$, L_n denotes the graph with 'n' vertices $\{V_1, V_2, V_3 \ldots V_n\}$ and with edges (V_i, V_{i+1}) for $1 \leq i \leq n$.

L_n ~~denotes~~ is called linear graph with 'n' vertices.

Ex:-  are linear graphs.

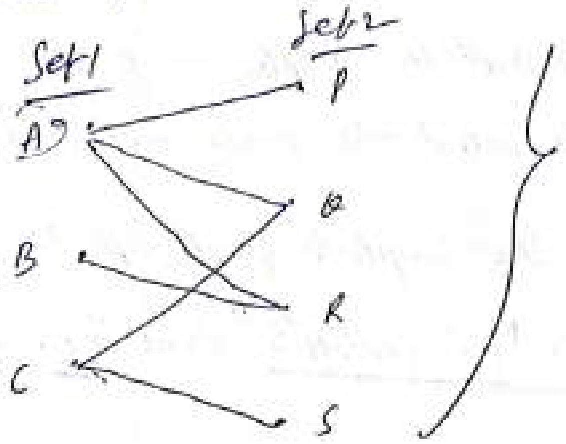
L_2 L_4

(4) Bipartite graph:- (partition graph)

A graph 'G' is called bipartite if its vertex set 'V' is partitioned into two disjoint subset of V_1 and V_2. Such that every edge in 'G' joins a vertex in V_1 with a vertex in V_2.

The graph is denoted by $G(V_1, V_2; E)$.

V_1 and V_2 is called bipartitions of the vertex set V.



Ex:

In this graph G, which vertex set is $V = \{A, B, C, P, Q, R, S\}$ and the edge set is

$E = \{AP, AQ, AR, BR, CQ, CS\}$.

V is the union of two of its subsets.

$V_1 = \{A, B, C\}$

$V_2 = \{P, Q, R, S\}$.

which are such that

1. V_1 and V_2 are disjoint.
2. every edge in G is joins of vertex in V_1 & V_2.
3. G contains no edge that joins two vertices both of which are in V_1 or V_2.

This graph is a ~~bipartite~~ bipartite graph with

$V_1 = \{A, B, C\}$

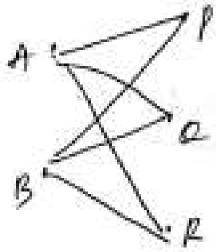
$V_2 = \{P, Q, R, S\}$ as bipartities.

(5) Complete Bipartite graph

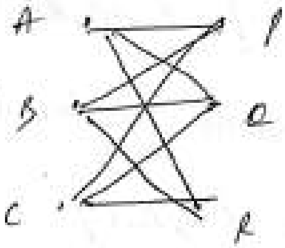
It is a bipartite graph in which every vertex in V_1 is adjacent to each vertex in V_2.

$K_{m,n}$ denotes the complete ~~graph~~ bipartite graph with "V_1 contains 'm' vertices" and "V_2 contains 'n' vertices".

Ex:- $K(2,3)$



$K(3,3)$

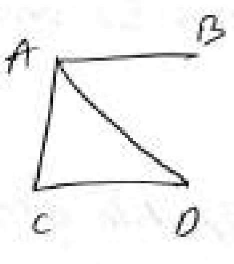


this is known as ~~Kuratow~~ Kuratowski's second graph.

Ex: Draw the diagram of the graph $G = (V,E)$ in each of the following cases. Find these graphs are connected or not?

(1) $V = \{A, B, C, D\}$ $E = \{(A,B), (A,C), (A,D), (C,D)\}$.

Sol:-

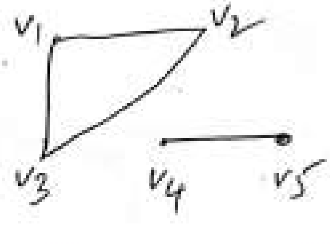


$\Rightarrow$ It is a connected graph.

(2) $V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{(v_1,v_2), (v_1,v_3), (v_2,v_3), (v_4,v_5)\}$.

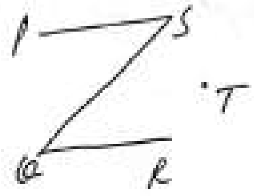
Sol:-



$\Rightarrow$ disconnected graph.

(3) $V = \{P, Q, R, S, T\}$

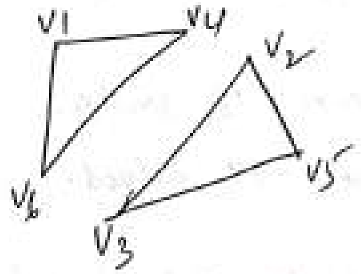
$E = \{(P, S), (Q, R), (Q, S)\}$

Sol:- 

$\Rightarrow$ disconnected graph.

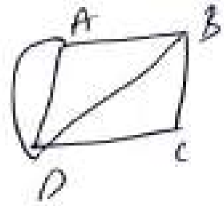
(4) $V = \{V_1, V_2, V_3, V_4, V_5, V_6\}$

$E = \{(V_1, V_4), (V_1, V_6), (V_4, V_6), (V_3, V_2), (V_3, V_5), (V_2, V_5)\}$.

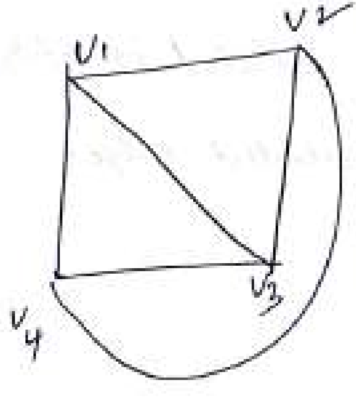
Sol:- 

$\Rightarrow$ disconnected graph.

Ex: which of the followings are the complete graphs.

(1) 

$\Rightarrow$ It is not a complete graph becoz there is no edge between A & C.

(2) 

$\Rightarrow$ This graph is complete. It is simple and there is a edge between every two vertices.

Ex:- How many vertices and how many edges are there in complete bipartite graphs $K_{4,7}$ and $K_{7,11}$?

Note:- The complete bipartite graph $K_{r,s}$ has "$r+s$" vertices and "rs" edges.

Sol:- according to that.

(i) The graph $K_{4,7}$ has $4+7=11$ vertices.
$4 \times 7 = 28$ edges.

the graph $K_{7,11}$ has $7+11=18$ vertices
$7 \times 11 = 77$ edges.

(ii) If the graph $K_{r,12}$ has 72 edges. what is r?

$$r \times 12 = 72$$

$$r = \frac{72}{12} = 6.$$

Isolated Graph:

Definition: A vertex is said to be isolated if its degree is zero. or it has no incident edge.

Subgraphs:

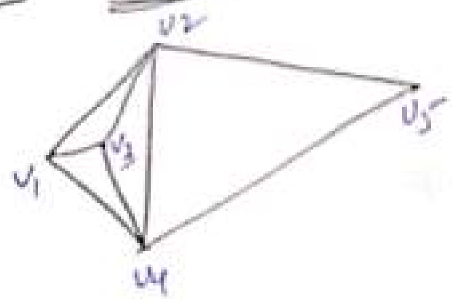
let $G(V,E)$ be a graph, the $H(V',E')$ obtained by deleting few vertices and edges from 'G' is called a subgraph of G. provided vertices V' in H contains all the end points of edges in E'.

Mathematically Stated

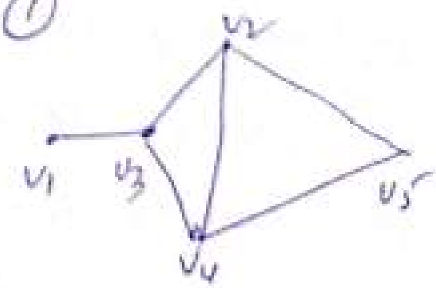
Definition: A Graph $H(V',E')$ is the subgraph of the graph $G(V,E)$, if vertices and edges of H are contained in vertices and edges of G.

i.e $V' \subseteq V$ and $E' \subseteq E$.

Ex: Graph $G(V,E)$. Subgraph $H(V',E')$.

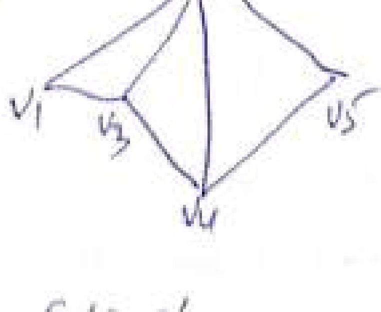


(1)



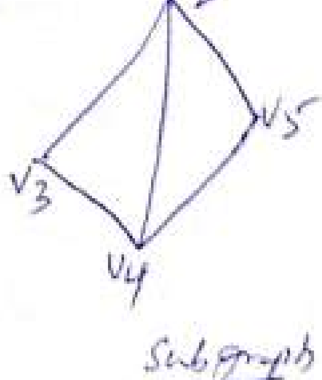
Subgraph

(2)



Subgraph

(3)

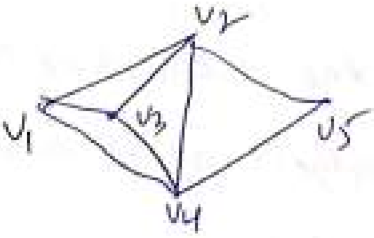


Subgraph

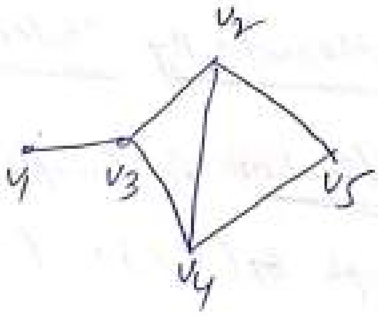
Spanning Subgraph

Definition: A Graph H is called a Spanning Subgraph of given graph G. if H contains ($V' = V$) all vertices of G.

Ex: Graph



Spanning Subgraph



Operations on Graphs

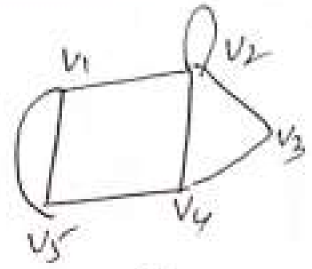
There are five operations performed on graphs.

1. Deleting a vertex.
2. Deleting an edge.
3. Complement of a graph.
4. Union of two graphs.
5. Intersection of two graphs.
6. Fusion.

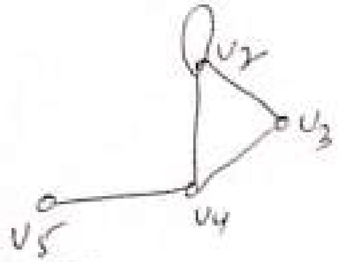
1. Deleting a vertex:

Let G be a graph with $V(G) = \{v_1, v_2, v_3, v_4 \ldots v_n\}$ then "$G - v_i$" is the graph obtained deleting or removing the vertex v_i from 'G' together with all edges incident on v_i.

Ex:-



G

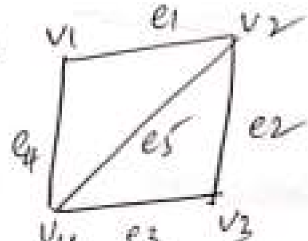


G - v1

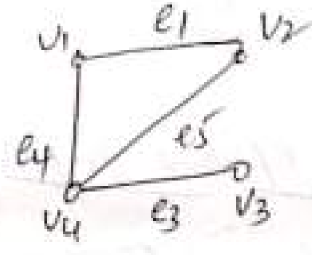
② Deleting an edge:

let G be a graph with E(G) = {e1, e2 --- en} then "G - e1" is the graph obtained by removing or deleting the edge ei without deleting the vertices which are incidented with ei.

Ex:



G



G - e2

③ Complement of a graph:

The complement of a graph 'G' is the graph $\bar{G}$ with the same vertices as G.

An edge exists in $\bar{G}$ ⟺ it does not exist in G, in otherwords, two vertices adjacent in G' ⟺ they are not adjacent in G. i.e

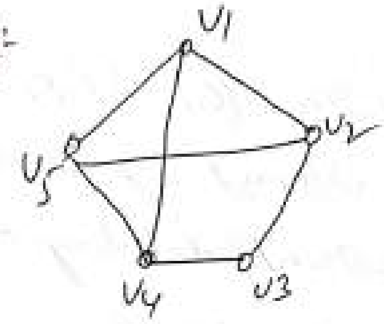
$\bar{G}(V, \bar{E})$ where $\bar{E} = V \times V - E$.

~~① Graph~~

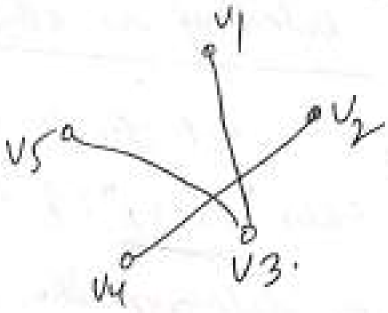
~~② ... graph~~

~~③ regular graphs.~~

Ex:-



G



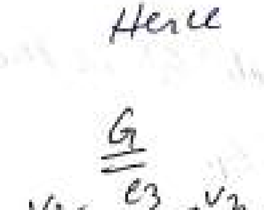
G' or $\bar{G}$

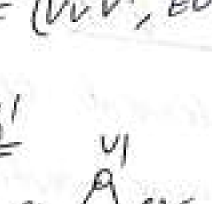
④ **union of two graphs**

Definition: let G and G' be two graphs the union of G and G' is the graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$.

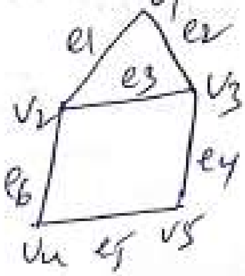
Here $G \cup G' = (V \cup V', E \cup E')$

Ex: G G'





$G \cup G'$:



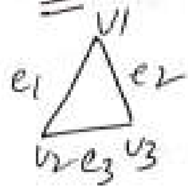
(5) Intersection of two graphs

The intersection of G_1 and G_2 is the graph consisting only those vertices and edges that are both in G_1 and G_2.

$$\Rightarrow G_1 \cap G_2 = (V_1 \cap V_2,\ E_1 \cap E_2)$$

Ex:- G_1

G_2



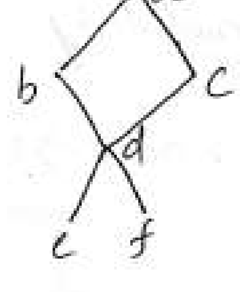
$G_1 \cap G_2$:-

(6) Fusion:

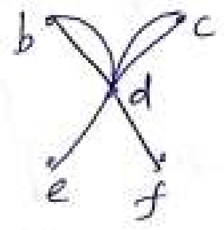
Fusion of two vertices 'a' and 'b' in a graph G is an operation G on which two vertices a and b are fused (merged) together without deletion of any edge of G.

Ex:

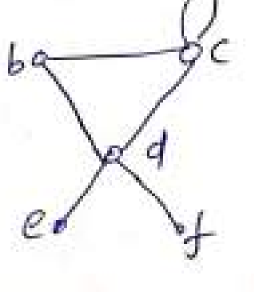
G



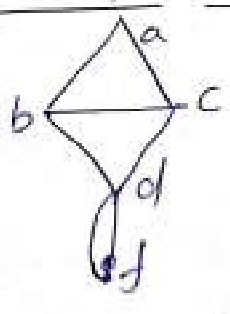
(a) Fusion of the vertices 'a' and 'b'



(b) Fusion of the vertices 'a' and 'c'



(c) Fusion of vertices e & f



Matrix Representation of graphs

Matrix Representation:

1. A diagrammatic representation of graph has limited usefullness.
2. further, such a representation is only possible when the no: of nodes and ~~edges~~ edges is reasonably small.
3. ~~A~~ A matrix is a convenient and useful way of representing a graph.
4. It is easy to store and manipulate matrices in a computer. Hence matrix representation of graphs has several advantages.

There are two types of matrix Representation

1. Adjacency Matrix
2. Incidence Matrix.

1. Adjacency matrix:

Def:- let $G=(V,E)$ be a simple graph with 'n' vertices then an $n \times n$ matrix is defined as

$$A = [a_{ij}]_{n \times n} \text{ is denoted by}$$

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge.} \\ 0 & \text{otherwise.} \end{cases}$$

Properties of adjacency matrix (of undirected simple graph)

① An Adjacency matrix completely defines a simple graph

② The Adjacency matrix is symmetric.

③ Any element of the adjacency matrix is either '0' or '1' therefore it is also called as, bit matrix (or) boolean matrix.

④ Adjacency matrix depends on the ordering of vertices.

⑤ It is a vertex - vertex adjacency matrix.

⑥ The i^{th} row in the adjacency matrix is determined by the edges which originate in the node v_i.

⑦ If the graph 'G' is simple, the degree of the vertex v_i equals the no. of 1's in the i^{th} row or (i^{th} column) of A_G.

Note:

In case of Directed graph

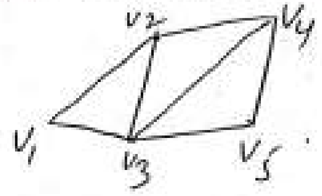
let $G(V,E)$ be a simple digraph with $V = \{v_1, v_2 \dots v_n\}$ and the vertices are assumed to be ordered from v_1 to v_n.

An $n \times n$ matrix A whose elements a_{ij} are given by

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

is called the adjacency matrix of the graph.

Ex:- Find out the matrix representation of following graph?



Sol: It is the 5x5 matrix. consisting of 5 vertices ordered v_1, v_2, v_3, v_4, v_5.

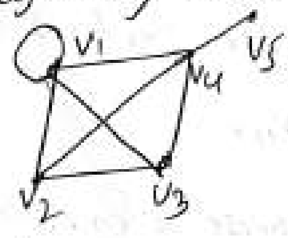
And each (v_i, v_j) of G is represented twice by

$$a_{ij} = 1 \text{ and } a_{ji} = 1.$$

$$A = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 1 & 0 & 0 \\ v_2 & 1 & 0 & 1 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 & 1 \\ v_4 & 0 & 1 & 1 & 0 & 1 \\ v_5 & 0 & 0 & 1 & 1 & 0 \end{array}$$

Note: the no. of non-zero elements in the matrix is equal to the sum of degrees of all vertices of graph.

Ex: write adjacency matrix for the graph, as shown below



Sol:

$$\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 1 & 1 & 1 & 1 & 0 \\ v_2 & 1 & 0 & 1 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 & 0 \\ v_4 & 1 & 1 & 1 & 0 & 1 \\ v_5 & 0 & 0 & 0 & 1 & 0 \end{array}$$

(2) Incidence Matrix

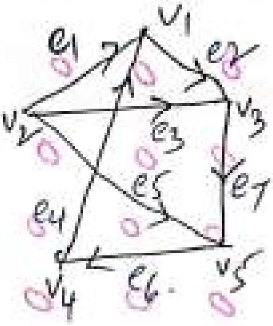
For a directed graph 'G' consists of 'n' vertices and 'm' edges, an $n \times m$ incidence matrix $M = [m_{ij}]$ is defined as.

$$M_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is initial vertex of edge } e_j \\ -1 & \text{if } v_i \text{ is final vertex of edge } e_j \\ 0 & \text{if } v_i \text{ is not incident on edge } e_j. \end{cases}$$

Note: The non-zero elements in the matrix is equal to the no: of edges of each vertex in the digraph.

$\Rightarrow$ if it is undirected graph all -1 entries are '0'.

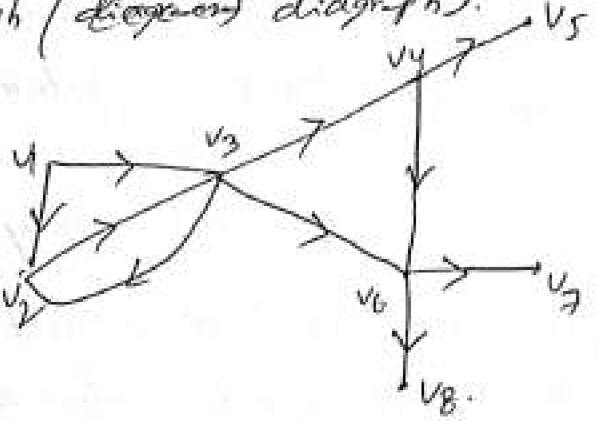
Ex: find the incidence matrix of the following digraph.



sol: Incidence matrix

$$M = \begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline v_1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \\ v_2 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ v_3 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ v_4 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ v_5 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{array}$$

Ex: Write the adjacency matrix for the following directed graph (~~diagram~~ diagraph).



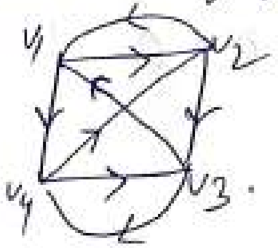
Sol:

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
V_1	0	1	1	0	0	0	0	0
V_2	0	0	1	0	0	0	0	0
V_3	0	1	0	1	0	1	0	0
V_4	0	0	0	0	1	1	0	0
V_5	0	0	0	0	0	0	0	0
V_6	0	0	0	0	0	0	1	1
V_7	0	0	0	0	0	0	0	0
V_8	0	0	0	0	0	0	0	0.

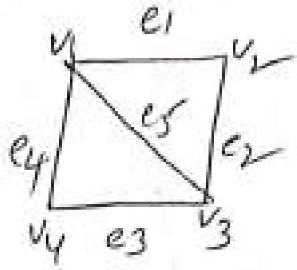
Ex: Draw the diagraph with the given matrix as adjacency matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Sol: The required diagraph is.



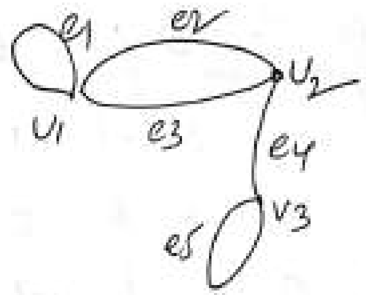
Ex: Find the incidence matrix of following graph.



Sol: Incidence Matrix is

	e_1	e_2	e_3	e_4	e_5
v_1	1	0	0	1	1
v_2	1	1	0	0	0
v_3	0	1	1	0	0
v_4	0	0	1	1	0

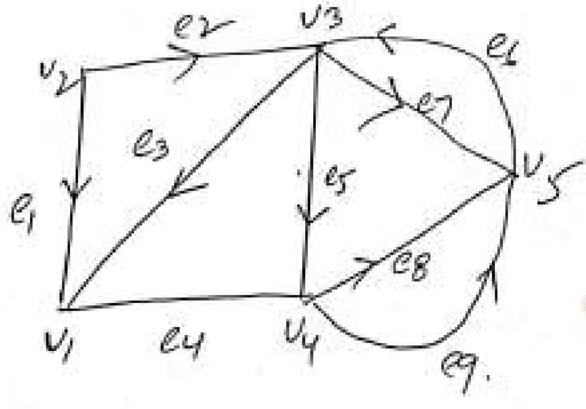
Ex: Find the incidence matrix of the following graph.



Sol:

	e_1	e_2	e_3	e_4	e_5
v_1	1	1	1	0	0
v_2	0	1	1	1	0
v_3	0	0	0	1	1.

Ex: find the incidence matrix for the following Digraph?



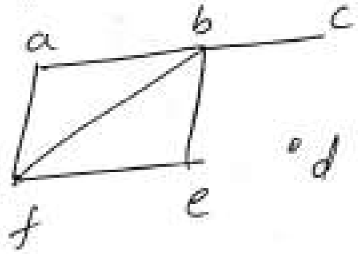
Sol:

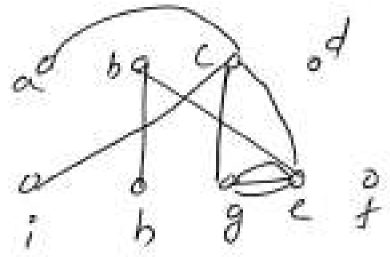
Incidence matrix (M) =

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
v_1	-1	0	-1	-1	0	0	0	0	0
v_2	1	1	0	0	0	0	0	0	0
v_3	0	-1	1	0	1	-1	1	0	0
v_4	0	0	0	1	-1	0	0	1	-1
v_5	0	0	0	0	0	1	-1	-1	1

Some examples on graphs (exercise problems)

① Find the no: of vertices, no: of edges and the degree of each vertex in the given undirected graph. Identify all isolated and pendent vertices.

ⓐ


ⓑ


Sol: (a) no: of vertices (V) = 6

no: of edges (E) = 6

deg (a) = 2

deg (b) = 4

deg (c) = 1

deg (d) = 0

deg (e) = 2

deg (f) = 3.

- pendant vertex is : 'c'.

isolated vertex is: 'd'.

(b) no: of vertices (V) = 9

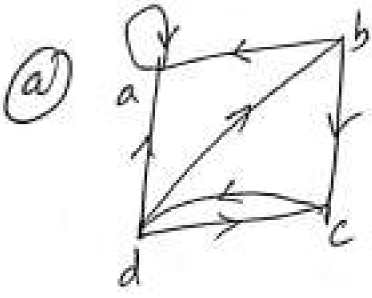
no: of Edges (E) = 12

deg (a) = 3	deg (e) = 6	deg (i) = 3.
deg (b) = 2	deg (f) = 0	deg i
deg (c) = 4	deg (g) = 4	
deg (d) = 0	deg (h) = 2	

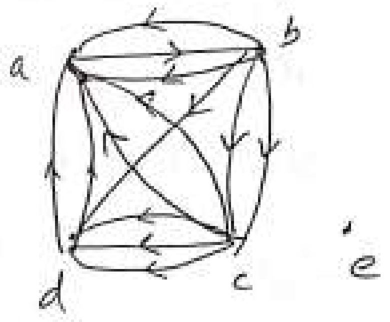
isolated vertices : d and f.

(2) Determine the no: of vertices and edges and find the Indegree and outdegree of each vertex for the given directed multi graph.

(a)



(b)



Sol: (a) no: of vertices = 4

no: of edges = 7

indeg(a) = 3	outdeg(a) = 1
indeg(b) = 1	out deg(b) = 2
indeg(c) = 2	out deg(c) = 1
indeg(d) = 1	out deg(d) = 3

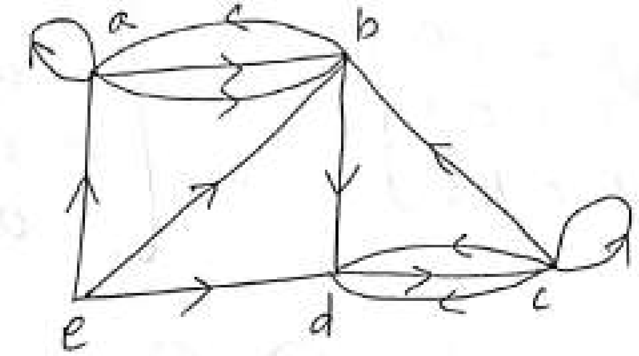
~~indeg~~

(b) no: of vertices = 4

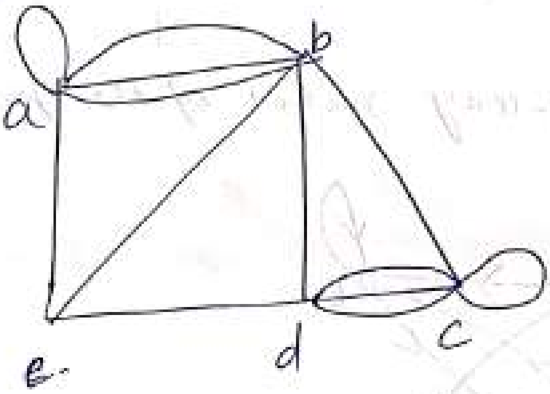
no: of Edges = 13.

indeg (a) = 6	outdeg (a) = 1
indeg (b) = 1	outdeg (b) = 5
indeg (c) = 2	outdeg (c) = 5
indeg (d) = 4	outdeg (d) = 2
indeg (e) = 0	outdeg (e) = 0.

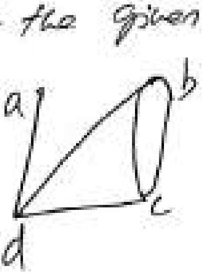
3) **Ex:-** construct the underlying undirected graph for the graph with directed edges for the following graph.

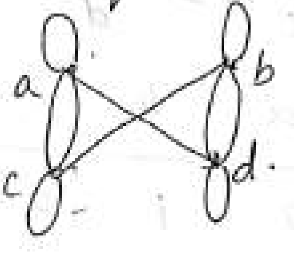


Sol:- Underlying undirected graph means we can draw the graph without Directions.



4) Represent the given graph using an adjacency matrix.

(a) 

(b) 

Sol:- (a)

	a	b	c	d
a	0	0	1	0
b	0	0	2	01
c	1	2	0	1
d	01	1	1	0

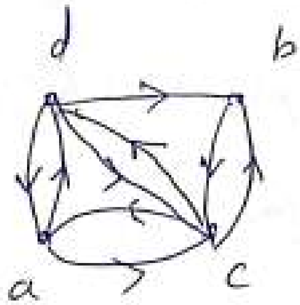
(b)

	a	b	c	d
a	1	0	2	1
b	0	1	1	2
c	2	1	1	0
d	1	2	0	1.

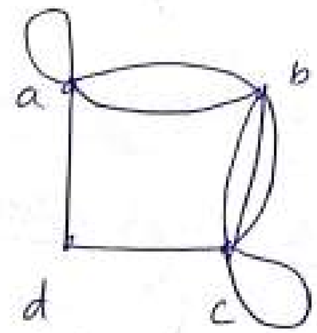
(5) Draw the graphs with given adjacency matrix.

(a) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

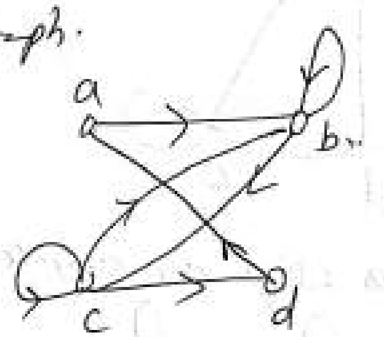
Sol: (a)



(b)



(6) Find the adjacency matrix of the given directed multi graph.



Sol:

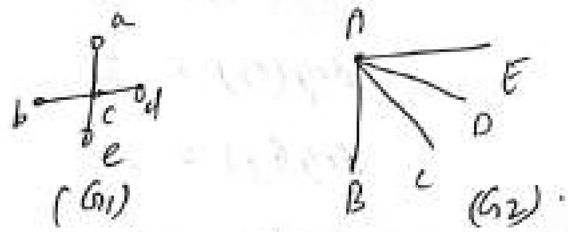
	a	b	c	d
a	0	1	0	0
b	0	1	1	0
c	0	1	1	1
d	1	0	0	0

Isomorphic Graphs:

let G_1 and G_2 i.e $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. These two graphs are said to be "isomorphic" if there exist a one-to-one and on-to correspondance between their vertices and as well as edges.

That is both graphs have equal no: of "vertices" and equal no: of "edges" and "degree sequence" is same.

Ex: Show that the following two graphs are isomorphic or not.

① 

Sol: G_1 — no: of vertices = 5 G_2 — no: of vertices = 5

no: of edges = 4 no: of edges = 4

deg(a) = 1 deg(A) = 4

deg(b) = 1 deg(B) = 1

deg(c) = 4 deg(C) = 1

deg(d) = 1 deg(D) = 1

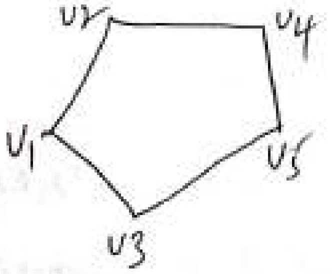
deg(e) = 1 deg(E) = 1

Degree sequence = 4 1 1 1 1 Degree sequence: 4 1 1 1 1.

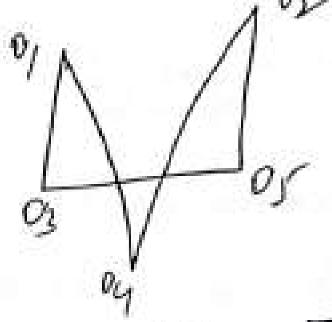
The above two graphs have equal no: of vertices, equal no: of edges and degree sequence is same.

∴ both two graphs are "isomorphic".

(2)



$\underline{\underline{G1}}$



$\underline{\underline{G2}}$

	o1	o2	o3	o4	o5
o1					
o2					
o3					
o4					
o5					

Sol: $\underline{\underline{G_1}}$ | $\underline{\underline{G_2}}$

no: of vertices = 5 | no: of vertices = 5

no: of Edges = 5 | no: of Edges = 5

$deg(v_1) = 2$ | $deg(o_1) = 2$

$deg(v_2) = 2$ | $deg(o_2) = 2$

$deg(v_3) = 2$ | $deg(o_3) = 2$

$deg(v_4) = 2$ | $deg(o_4) = 2$

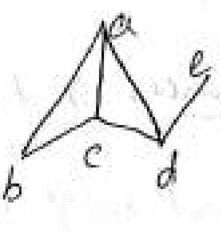
$deg(v_5) = 2$ | $deg(o_5) = 2$

degree sequence : 2 2 2 2 2 | degree sequence : 2 2 2 2 2.

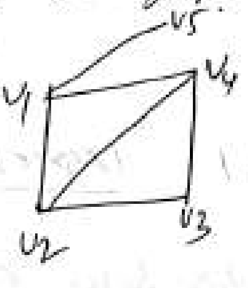
The above two graphs have equal no: of vertices, equal no: of edges and degree sequence is same.

Therefore both graphs are isomorphic.

(3)



$\underline{\underline{G_1}}$



$\underline{\underline{G_2}}$

	v_1	v_2	v_3	v_4	v_5
v_1	0	1	1	0	0
v_2	1	0	0	1	0
v_3	1	0	0	0	1
v_4	0	1	0	0	1
v_5	0	0	1	1	0

Sol: G_1 G_2

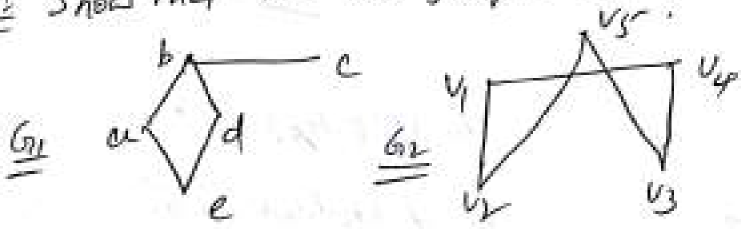
G_1	G_2
no: of vertices = 5	no: of vertices = 5
no: of Edges = 6	no: of Edges = 6
deg (a) = 3	deg (v_1) = 3
deg (b) = 2	deg (v_2) = 3
deg (c) = 3	deg (v_3) = 2
deg (d) = 3	deg (v_4) = 3
deg (e) = 1	deg (v_5) = 1

- degree sequence: 1 2 3 3 3   Degree sequence: 1 2 3 3 3.

The above two graph have equal no: of vertices, equal no: of edges and degree sequence is same.

There fore both graphs are "isomorphic".

Ex: Show that the two graphs are isomorphic or not.

- 

Sol: G_1 G_2

G_1	G_2
no: of vertices = 5	no: of vertices = 5
no: of Edges = 5	no: of Edges = 5
deg (a) = 2	deg (v_1) = 2
deg (b) = 3	deg (v_2) = 2
deg (c) = 2	deg (v_3) = 2
deg (d) = 2	deg (v_4) = 2
deg (e) = 2	deg (v_5) = 2.

Degree Sequence of G_1 Degree Sequence of G_2

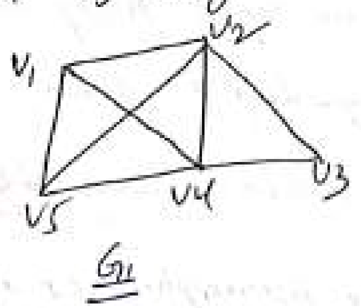
2 2 2 2 3 2 2 2 2 2 .

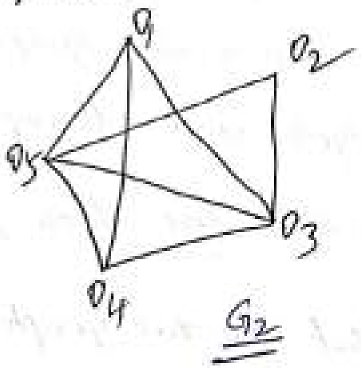
~~Ele~~ ~~G1~~ ~~&~~ both graphs have equal no: of vertices, equal no: of edges ~~Etd~~ but degree sequence is not same.

∴ two graphs are "not Isomorphic".

⇒ Another way to prove two graphs are isomorphic is "matrix representation".

⇒ determine the following pairs of graphs are isomorphic or not by using matrix representation.





Sol:- G_1

no: of vertices = 5

no: of Edges = 8

no: of Edges = 8

no: of vertices = 5

deg(v_1) = 3

deg(v_2) = 4

deg(v_3) = 2

deg(v_4) = 4

deg(v_5) = 3

deg(o_1) = 3

deg(o_2) = 2

deg(o_3) = 4

deg(o_4) = 3

deg(o_5) = 4

One-to-one correspondence between G_1 & G_2.

$deg(v_1)$

degree 4 $\Rightarrow deg(v_2) = deg(o_3)$.

$deg(v_4) = deg(o_5)$.

degree 3 $\Rightarrow deg(v_1) = deg(o_1)$

$deg(v_5) = deg(o_4)$.

degree 2 $\Rightarrow deg(v_3) = deg(o_2)$.

i.e. $deg(v_1) = o_1$

$deg(v_2) = o_3$

$deg(v_3) = o_2$

$deg(v_4) = o_5$

$deg(v_5) = o_4$.

matrix for G_1

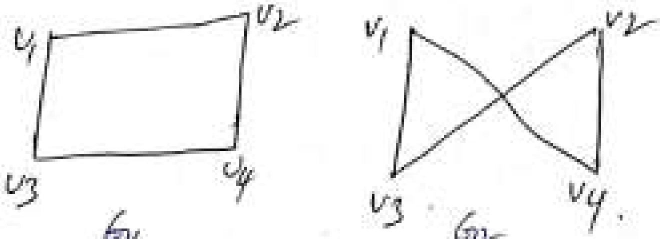
	v_1	v_2	v_3	v_4	v_5
v_1	0	1	0	1	1
v_2	1	0	1	1	1
v_3	0	1	0	1	0
v_4	1	1	1	0	1
v_5	1	1	0	1	0

matrix for G_2

	o_1	o_3	o_2	o_5	o_4
o_1	0	1	0	1	1
o_3	1	0	1	1	1
o_2	0	1	0	1	0
o_5	1	1	1	0	1
o_4	1	1	0	1	0

Since the two adjacency matrices are the same, the two graphs are isomorphic.

Ex: Show that the following graphs G and G' are isomorphic?



G_1 $\qquad\qquad$ G_2

Sol: The two graphs have the same no: of vertices, same number of edges and degree sequence.

G_1	G_2
no: of vertices = 4	no: of vertices = 4
no: of Edges = 4	no: of Edges = 4
$\deg(u_1) = 2$	$\deg(v_1) = 2$
$\deg(u_2) = 2$	$\deg(v_2) = 2$
$\deg(u_3) = 2$	$\deg(v_3) = 2$
$\deg(u_4) = 2$	$\deg(v_4) = 2$

now define the function 'f' as follows.

$$f(u_1) = v_1, \quad f(u_2) = v_4, \quad f(u_3) = v_3, \quad f(u_4) = v_2$$

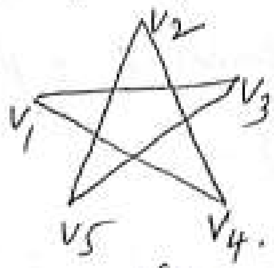
$$A(G) = \begin{array}{c|cccc} & u_1 & u_4 & u_3 & u_2 \\ \hline u_1 & 0 & 1 & 1 & 0 \\ u_4 & 1 & 0 & 0 & 1 \\ u_3 & 1 & 0 & 0 & 1 \\ u_2 & 0 & 1 & 1 & 0 \end{array}$$

$$A'(G') = \begin{array}{c|cccc} & v_1 & v_4 & v_3 & v_2 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_4 & 1 & 0 & 0 & 1 \\ v_3 & 1 & 0 & 0 & 1 \\ v_2 & 0 & 1 & 1 & 0 \end{array}$$

$\Rightarrow$ f is one to one and onto also it is preserving the adjacency of the vertices. $\therefore$ Two graphs are isomorphic to each other.

Ex: Determine whether the given pair of graphs are isomorphic.

G_1



G_2

Sol:

G_1	G_2
no: of vertices = 5	no: of vertices = 5
no: of edges = 5	no: of edges = 5
$deg(v_1) = 2$	$deg(v_1) = 2$
$deg(v_2) = 2$	$deg(v_2) = 2$
$deg(v_3) = 2$	$deg(v_3) = 2$
$deg(v_4) = 2$	$deg(v_4) = 2$
$deg(v_5) = 2$	$deg(v_5) = 2$

no: of vertices, edges and degree sequence is same.

$\therefore$ the two graphs are isomorphic.

Tree:

A tree is a connected graph without cycles or loops.

properties of the tree:

① A tree is a connected and there exist a unique path from the root to every other vertex.

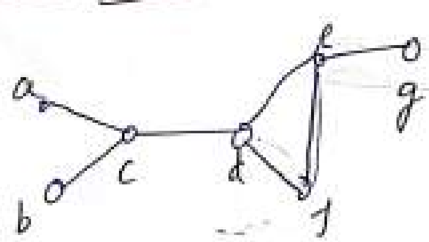
② A tree contains 'n' vertices and "n-1" edges.

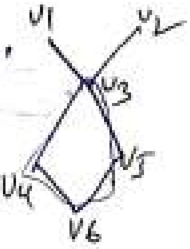
③ If the outdegree of every node in a tree is $\leq m$, then such tree is called an m-ary tree.

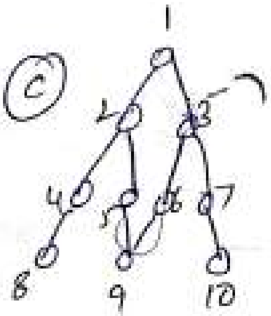
If the out ~~degre~~ degree of every node is equal to '0' or 'm' then such tree is called a complete m-ary tree.

In this case $m=2$ is most important.

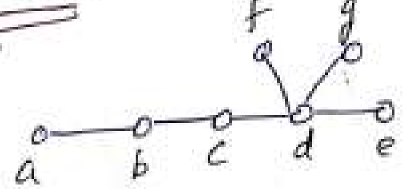
Ex: Not a tree:

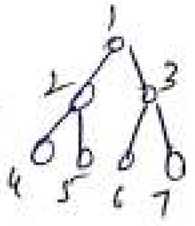
ⓐ


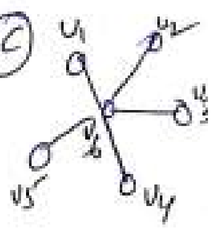
ⓑ


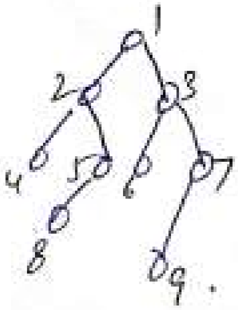
ⓒ


Trees:

ⓐ


ⓑ


ⓒ


ⓓ


4.20

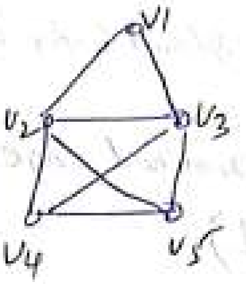
Spanning tree:

Definition: let 'G' be a connected graph. A subgraph T is called a spanning tree of "G".

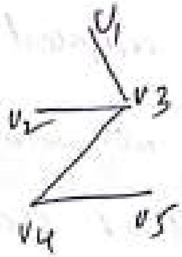
if ① T is a tree
② T contains all vertices of G.

Example:

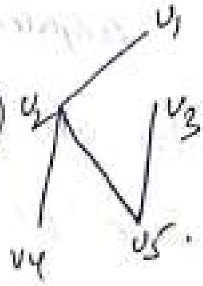
Graph G:



Spanning trees: ①



②



There are two alg to find the spanning tree of a connected graph G.

① Depth first search (DFS)
② Breadth first search (BFS).

DFS:

The idea of DFS is proceeding to higher levels successively in the first opportunity. Later we back and add the vertices which are not visited.

Following are various steps performed in DFS alg.

input: A connected graph G with vertices ~~which are~~ not visited. $v_1, v_2, v_3 - \cdots v_n$.

output: A spanning tree T for G.

Step 1:- (Visiting a vertex)

let v_1 be the root of 'T' and set $L = v_1$.

(L stands for the vertex last visited)

step 2:- (Find an unexamined edge and an unvisited vertex adjacent to L.)

For all vertices adjacent to L choose the edge $\{L, v_k\}$, where 'k' is the minimum index such that adding $\{L, v_k\}$ to T does not create a cycle.

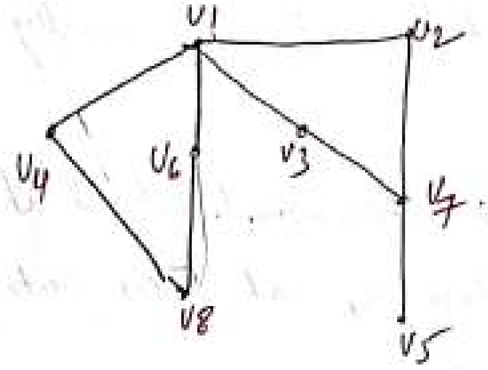
If no such edge exists goto step3. otherwise add edge $\{L, v_k\}$ to T and set $L = v_k$. repeat step2 at the new value for L.

Step 3:- (Back tracking or terminating)

If x is the parent of L in T, set $L = x$ and apply step2 at the new value of L.

If on the otherhand, L has no parent in T (so that $L = v_1$) then the DFS terminates and T is a spanning tree for G.

Ex: Illustrate DFS alg on the graph given below for finding spanning tree.



Sol:

→ Now we select the v_1 as the root of T. At this point vertex 'v_1' is visited.

→ now we should select the edge $[v_1, x]$ where x is the first label in the designated order and which doesn't form a cycle with the edges that are choosen in 'T' previously.

→ since edge $\{v_1, v_2\}$ does not form a cycle we select edge $[v_1, v_2]$. next we select $\{v_2, v_7\}$ since it does not form a cycle.

→ At the v_7 we can select either $\{v_7, v_3\}$ or $\{v_7, v_5\}$ But we select $\{v_7, v_3\}$ because it comes first.

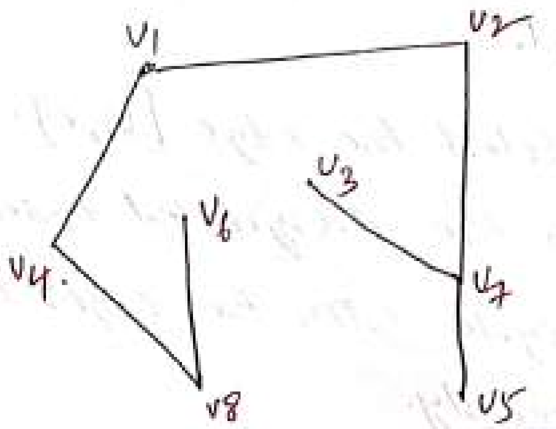
→ now at v_3 if we select $\{v_3, v_1\}$ a cycle gets formed so we come back to v_7. And find that there is one unexamined edge $\{v_7, v_5\}$ so we select it.

→ now at v_5 no more unexamined edges are there. so we back track to v_1 and at v_1 we select edge $\{v_1, v_4\}$

→ now at v_4 we select $\{v_4, v_8\}$ and finally we select $\{v_8, v_6\}$ edge.

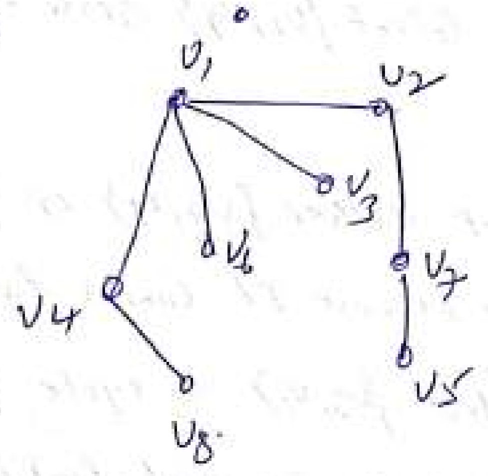
→ now there are more unexamined edges at v_6. we back track to v_8 then v_4 and so on until we come back to v_1. i.e root.

The below is the DFS spanning tree obtained.



Spanning tree T using DFS

BFS



② BFS algorithm

The idea of BFS is to visit all vertices sequentially on a given level before going onto the next level.

Input: A connected graph G with vertices labelled $v_1, v_2 \cdots v_n$.

output: A spanning tree T ~~for~~ of G.

followings are various steps performed in BFS algorithm

step1: let v_1 be the root of G. Form the set $V = \{v_1\}$.

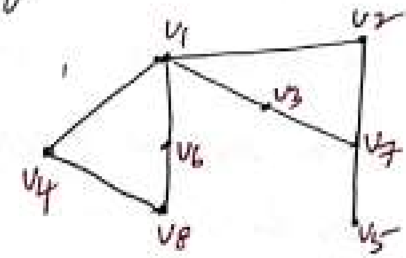
step2: Adding the new edges: consider the vertices of the graph in order, consistent with the original labelling, the for each vertex $x \in V$ add the new edge $\{x, v_k\}$ to T, where k is the minimum index such that adding the edge $\{x, v_k\}$ does not produce any cycle.

If no edge can be added then STOP. Then T is a spanning tree for G. After all the vertices of 'V' have been considered in order ~~to~~ goto step 3.

step3: Replace 'V' by all the children V in T of the vertices x of V where the edges $\{x, v\}$ were added in step2.

Go back and repeat step2 for the new set V.

Ex: Illustrate BFS alg on the graph given below for finding spanning tree.



Sol: Now we will find BFS spanning tree for the graph order of vertices are shown.

⇒ we select 'V' as first vertex and designated it has root of T.

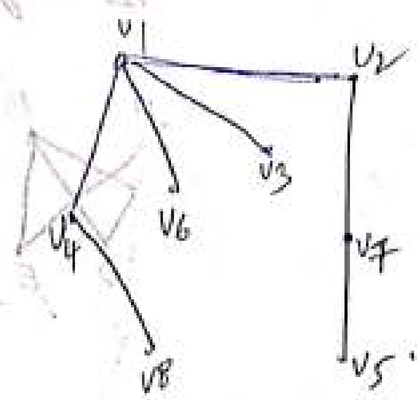
⇒ AT this point T contains a simple vertex V_1. Now added to T all edges $\{V_1, x\}$ as x run in order from V_1 to V_8. which do not produce a cycle.

⇒ we add $\{V_1, V_2\}, \{V_1, V_3\}, \{V_1, V_4\}, \{V_1, V_6\}$. These edges are called tree edges for BFS tree.

⇒ We must repeat the process for all vertices on level one from the root by examing each vertex in the designed order V_2, V_3, V_4 and V_6 are at level one.

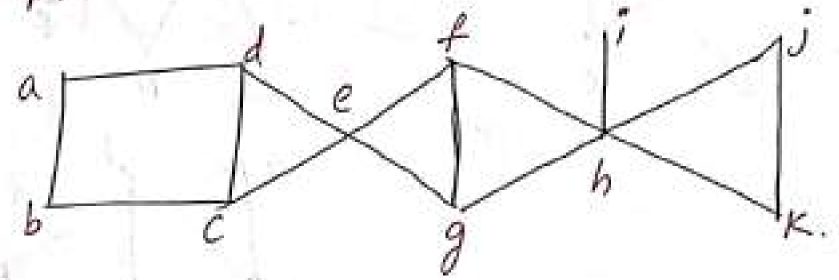
⇒ we select V_2 first to examine, we include $\{V_2, V_7\}$. Similarly we select $\{V_4, V_8\}$ but reject $\{V_6, V_8\}$ and $\{V_3, V_7\}$. Since they form a cycle and in the next step applying the same procedure. next we select $\{V_7, V_5\}$.

⇒ the edges which are rejected in BFS are known as cross edges. thus we finally obtain the BFS spanning tree as shown below.

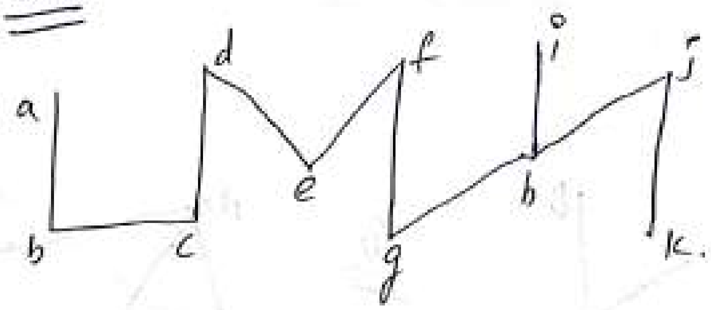


Spanning tree using BFS algorithm.

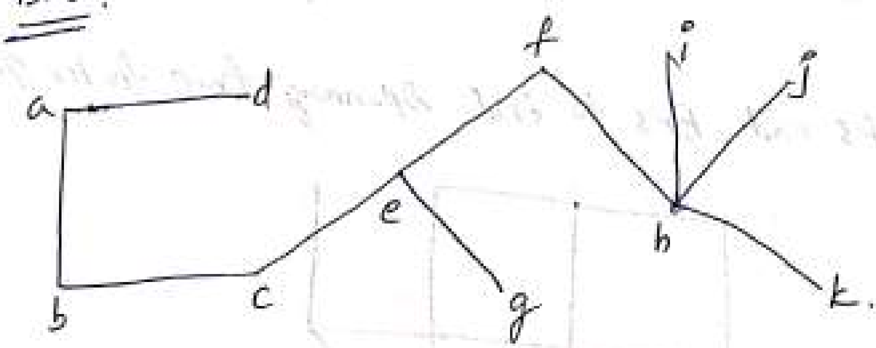
Ex: find the DFS and BFS spanning tree using following graph.



Sol:- DFS:

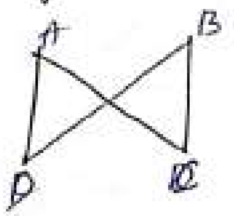


BFS:

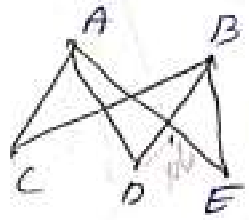


Ex: Draw all spanning trees of graph G_1, G_2 using DFS and BFS alg.

G_1

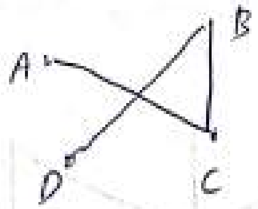


G_2

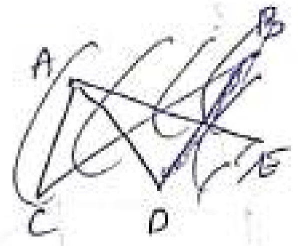


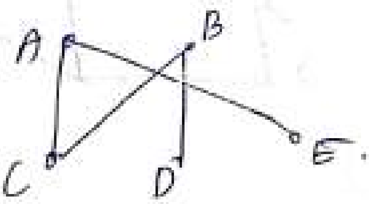
Sol: **DFS**

G_1



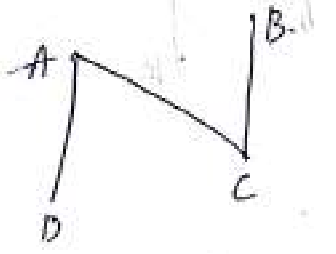
G_2



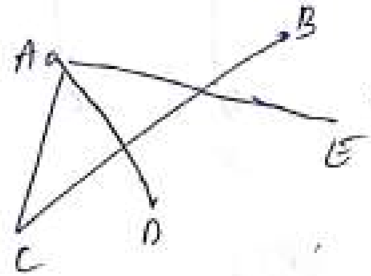


BFS:

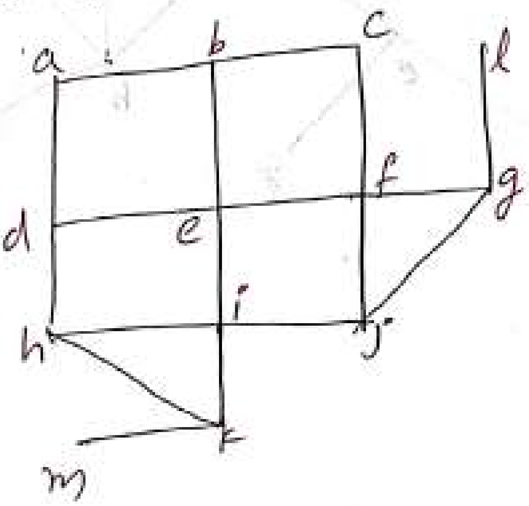
G_1



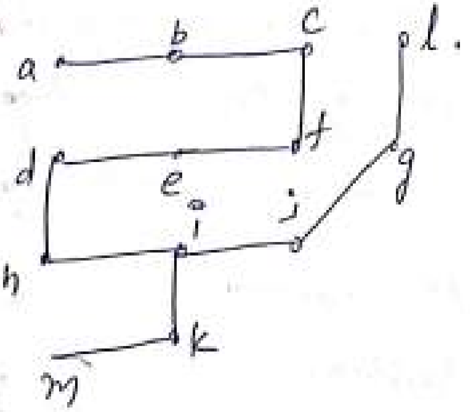
G_2



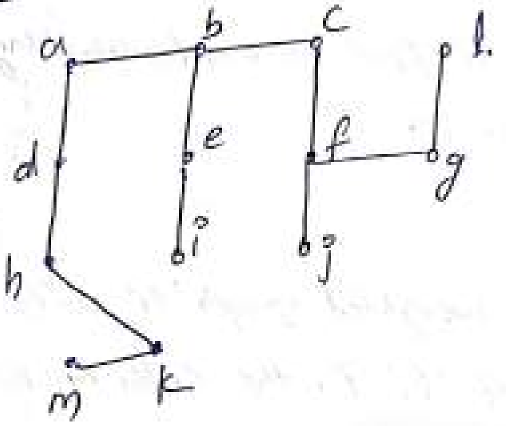
Ex: Use DFS and BFS to find spanning tree for the graph?



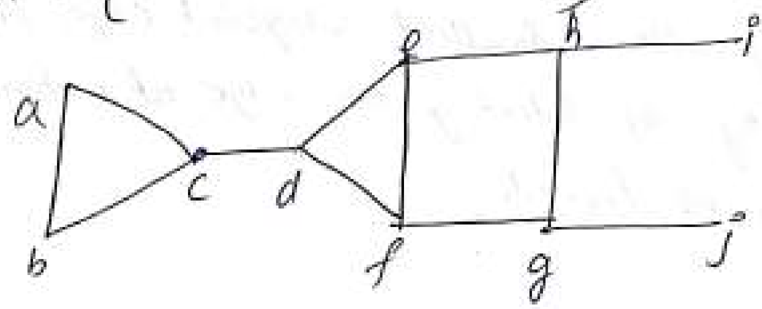
Sol: **DFS:**



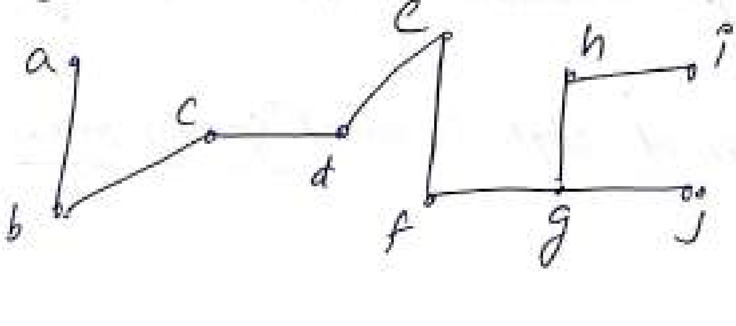
BFS



Ex: Use DFS to produce a spanning tree for the given simple graph G (Choose 'a' as the root).



Sol: **DFS**



Minimum Spanning tree:

Definition: A spanning tree whose weight is the least is called minimal spanning tree. this tree is not unique.

There are two wfy to find the minimum spanning tree.

(1) kruskal's algorithm.

(2) prim's algorithm.

(1) kruskal's algorithm:

The procedure for to find minimal spanning tree using kruskal's algorithm is

step1:

Given a connected weighted graph 'G' with 'n' vertices and list the edges of 'G' in the order of non-decreasing weights.

step2:

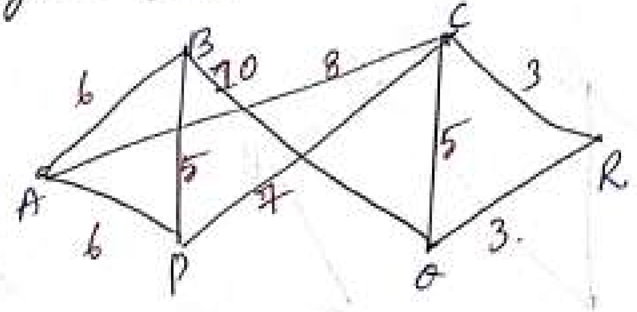
Starting with a smallest weighted edge, proceed sequentially by selecting one edge at a time such that no cycle is formed.

step3:

stop the process of step2 when 'n-1' edges are selected. these n-1 edges constitute a minimal spanning tree of G.

Note: The process of step2 is called "greedy process"

Ex: Using kruskal's alg. find a minimal spanning tree of the weighted graph G.



Sol: Given graph has 6 vertices.

Hence minimum spanning tree will have 5 edges.

⇒ First put the edges of graph G in the non-decreasing order of their weights.

⇒ And successively select 5 edges in such a way that no cycle is created.

weights:	3	3	5	5	6	6	7	8	10
Edges:	CR	QR	BP	CQ	AB	AP	CP	AC	BQ.
Select:	yes	yes	yes	no	yes	no	yes	—	—

Explanation:

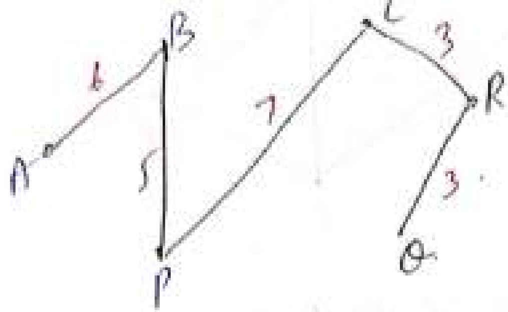
① CQ is not selected beco3 CR & QR have already been selected and the selection of CQ would have created a cycle.

② AP is not selected beco3 it would have created a cycle along with BP & AB.

which have already been selected.

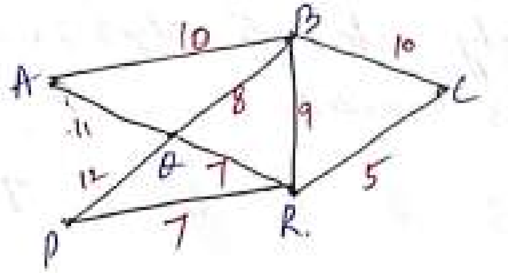
③ We have stopped the process when exactly 5 edges are selected.

The minimal spanning tree of the given graph contains the four edges CR, QR, BP, AB, CP.



The weights of this tree is = $\boxed{24 \text{ units}}$

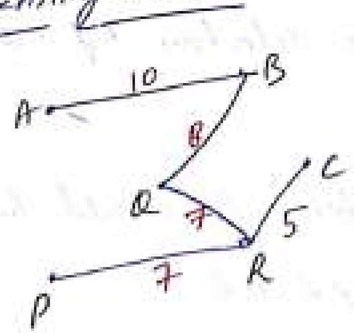
Ex:- Using Kruskal's alg, find the minimal spanning tree of the given weighted graph G.



Sol:- Edges

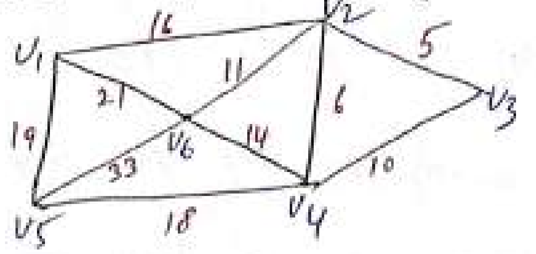
Weights	5	7	7	8	9	10	10	11	12
Edges	CR	PR	RQ	BQ	BR	BC	AB	AQ	PQ.
Select:	yes	yes	yes	yes	no	no	yes	—	—

Minimal spanning tree is



The weight of this spanning tree is = $\boxed{37 \text{ units}}$

Ex: Using kruskals to find MST (minimum spanning tree).



6 - vertices

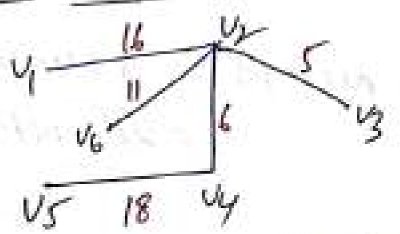
5 - edges are selected.

Sol:- Weights: 5 6 10 11 14 16 18 19 21 33.

Edges: v_2v_3 v_2v_4 v_3v_4 v_2v_6 v_4v_6 v_1v_2 v_4v_5 v_1v_5 v_1v_6 v_5v_6.

Select: yes yes no yes no yes yes — — —

minimum spanning tree is

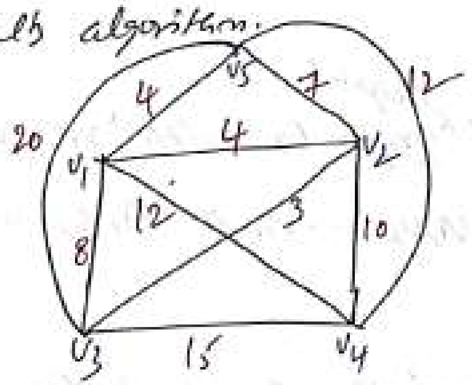


All the vertices of G are covered. therefore we stop the process.

minimal cost of MST = 5+6+11+16+18

$= \boxed{56 \text{ units.}}$

Ex: Give the MST of the following graph by using kruskals algorithm.



5 - vertices

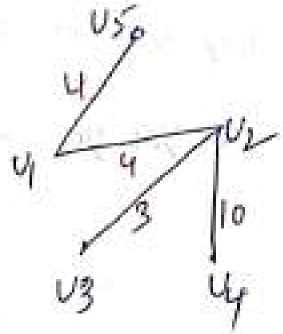
4 - edges are selected.

Sol;

Weights 3 4 4 7 8 10 12 12 15 20

Edges: v_1v_3 v_1v_2 v_1v_5 v_4v_5 v_1v_3 v_2v_4 v_1v_4 v_4v_5 v_3v_4 v_3v_5

Select: yes yes yes no no yes — — — —



minimal cost of MST is = 3+4+4+10

$= \boxed{21 \text{ units}}$

② Prim's alg!

TO find the minimal spanning tree using prims alg. we have two methods.

Ist Method Procedure: (Matrix representation).

Step1;

① Given a connected weighted graph G with 'n' vertices assign 'n' names (say $V_1, V_2 \text{---} V_n$ or A, B, C and so on) to these vertices.

② And prepare $n \times n$ table in which the weights of all edges are shown.

③ The entries in table will be symmetric with respect to the diagonal and no entries appear on the diagonal.

(4) Indicate the weights of the non existing edges as ∞.

Step2:

(1) Start from the vertex v_1 (or A_1) and connect it to its nearest neighbour (i.e to the vertex which has the smallest entry) in the v_1-row say v_k.

(2) now, consider the edge $\{v_1, v_k\}$ and connect it to its closest neighbour (i.e to a vertex other than v_1 and v_k)

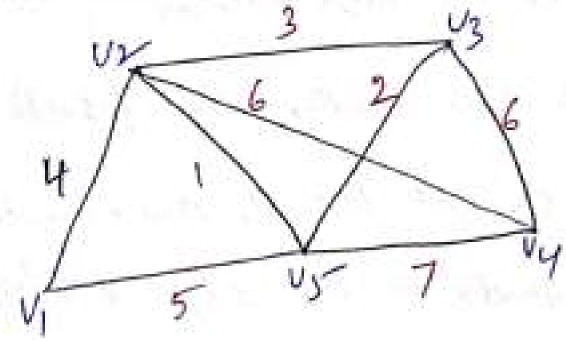
(3) That has the smallest entry among all entries in v_1 and v_k rows. let this vertex be v_m.

Step3:

(1) Start from the vertex v_m and repeat the process of step2. Stop the process when all the 'n' vertices have been connected by (n-1) edges.

(2) These (n-1) edges constitute a minimal spanning tree.

Ex: Using Prim's alg find minimal spanning tree for the weighted graph shown below.



Sol:- Graph has 5 vertices. minimal spanning tree have 4 edges.

Tabulate the weights of the edges between every pair of vertices.

	v_1	v_2 ×	v_3 ×	v_4	v_5 ×
✓ V_1	–	4	∞	∞	5
✓ V_2	4	–	3	6	1
✓ V_3	∞	3	–	6	2
V_4	∞	6	6	–	7
✓ V_5	5	1	2	7	–

(v_1, v_2) ✓
(v_2, v_5) ✓
(v_5, v_3)
(v_3, v_4)

Explanation:-

① Now start the first row (V_1-row) and pick the smallest entry there in. this 4, which corresponds to the edge $\{v_1, v_2\}$. By examining all the entries in v_1 & v_2.

② we find that vertex other than v_1 & v_2 which ~~correspo~~ corresponds to the smallest entry in v_5 (it being 1). thus v_5 is closest to the edge $\{v_1, v_2\}$. let us connect v_5 to the edge $\{v_1, v_2\}$ at v_2.

③ Becoz $\{v_5, v_2\}$ has smaller than ~~$\{v_1,v_5\}$~~ $\{v_2, v_5\}$.

④ let us now examine the v_5 row, smallest entry is 1 which corresponds to the edge is (v_5, v_2).

⑤ By examine all entries in v_5 & v_2. we find that vertex other than the v_2 & v_5 which corresponds to the smallest entry is v_3.

⑥ Thus v_3 is closest to the edge $\{v_2, v_5\}$. let us connect V_3 to the edge (v_2, v_5) at v_5.

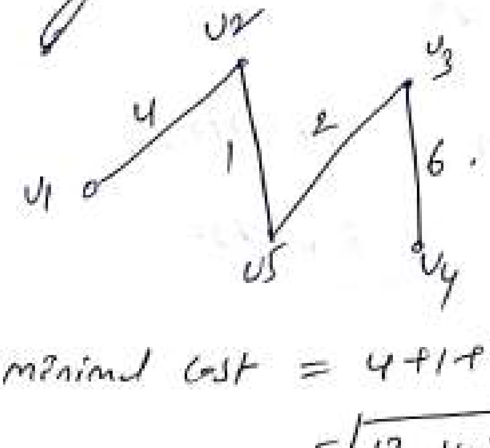
⑦ Thus the edges $\{v_1, v_2\}, \{v_2, v_5\}, \{v_5, v_3\}$ belongs to a minimal spanning tree.

⑧ The vertex left over at this stage is v_4. which joined to v_2, v_3, v_5 in the given graph among these edges that contains v_4.

⑨ The edges $\{v_4, v_2\}, \{v_4, v_3\}$ have equal minimum weight including these edges, the minimal spanning tree is

$\{v_1, v_2\}, \{v_2, v_5\}, \{v_5, v_3\}$ ~~or~~

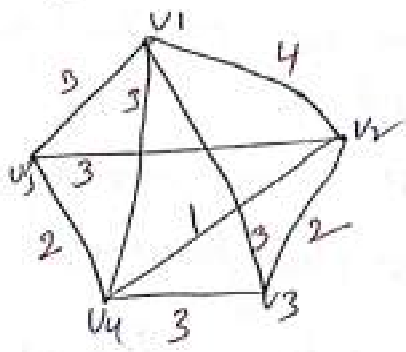
$\{v_4, v_3\}$



minimal cost $= 4+1+2+6$

$= \boxed{13 \text{ units}}$

Ex: use prim's alg find MST?

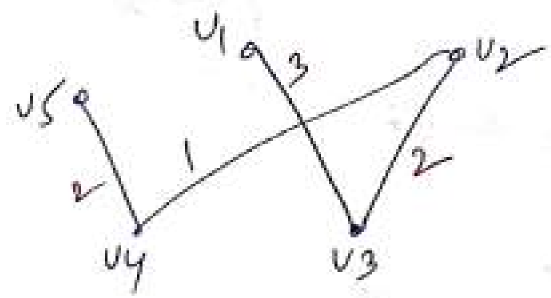


Sol:

	v_1	v_2	v_3	v_4	v_5	
✓ v_1	—	4	3	3	3	(v_1, v_3)
✓ v_2	4	—	2	1	3	(v_3, v_2)
✓ v_3	3	2	—	3	∞	
✓ v_4	3	1	3	—	2	(v_2, v_4)
v_5	3	3	∞	2	—	(v_4, v_5).

Selected edges are: $(v_1, v_3), (v_3, v_2), (v_2, v_4), (v_4, v_5)$.

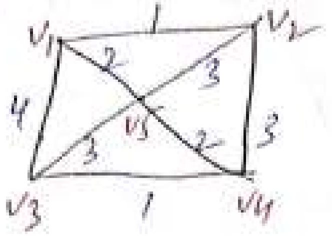
The minimal spanning tree is.



minimum cost is = $3+2+1+2$

$= \boxed{8 \text{ units.}}$

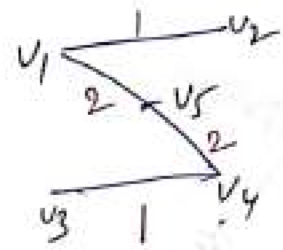
Ex:- using prim's find MST?



Sol:- n×n matrix is

	V1	V2	V3	V4	V5	
✓V1	–	1	4	∞	2	(V1,V2)
✓V2	1	–	∞	3	3	(V2,V4)
✓V3	4	∞	–	1	3	(V4,V3)
✓V4	∞	3	1	–	2	(V3,V5).
V5	2	3	3	2	–	

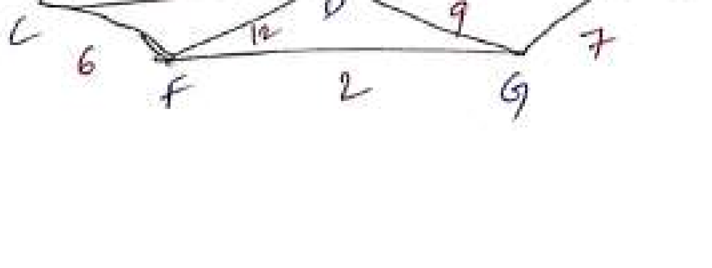
Selected edges are:- (V1,V2), (V2,V4), (V4,V3) & (V3,V5).



minimum cost is = 1+2+2+1

$= \boxed{6 \text{ units}}$

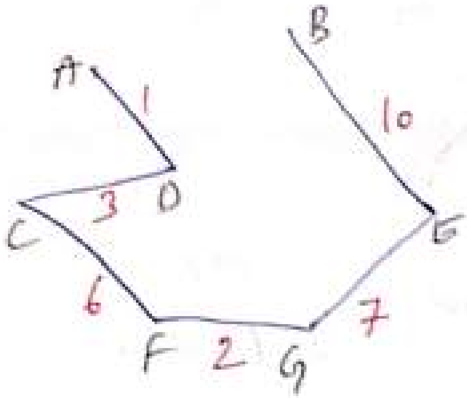
Ex:- using prim's alg to find a minimum spanning tree for given weighted graph G.



Sol:-

	A	B	C	D	E	F	G
A	—	5	4	1	∞	∞	∞
B	5	—	∞	14	10	∞	∞
C	4	∞	—	3	∞	6	∞
D	1	14	3	—	8	12	9
E	∞	10	∞	8	—	∞	7
F	∞	∞	6	12	∞	—	2
G	∞	∞	∞	9	7	2	—

Selected edges are: (A,D), (D,C), (C,F), (F,G), (G,E), (E,B).



minimum cost is $\Rightarrow 1+3+6+2+7+10$

$\Rightarrow$ $\boxed{\text{29 units}}$

IInd Method (prim's alg)

Step1:

Select an arbitrary vertex V_1, and an edge e_1 with minimum weight incident with vertex V_1.

this forms a initial MST for T.

step2:

If edges $e_1, e_2 \dots e_i$ have been choosen involving end points $V_1 V_2 \dots V_{i+1}$.

choose and edge $e_{i+1} = V_j V_k$ with $V_j \in T$ & $V_k \notin T$ such that e_{i+1} has smallest weight among the edges of G with one end in $\{V_1 \dots V_{i+1}\}$

step3:

stop after $(n-1)$ edges have been choosen. otherwise goto step2.

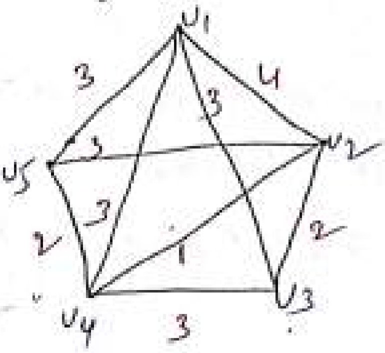
(OR)

Step1: Select any vertex and choose the edge of minimum weight from G.

Step2: At each state choose the edge of smallest weight joining a vertex already included to a vertex not yet included.

Step3: Continue ~~until~~ untill all vertices are included.

Ex: Find the minimal spanning tree of the weighted graph G using prim's algorithm.

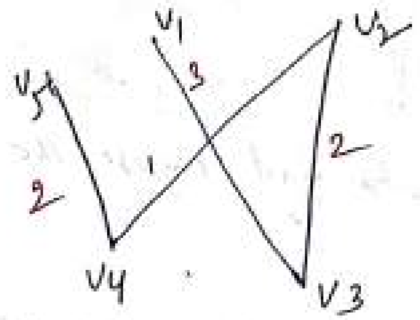


Sol: According to step 1, choose vertex v_1, now edge with smallest weight incident on v_1 is

$$e = v_1v_3 \text{ or } v_1v_5 \text{ or } v_1v_4$$

Choose $e = v_1v_3$ similarly chose the edges v_3v_2, v_2v_4, v_4v_5.

The minimal spanning tree is.

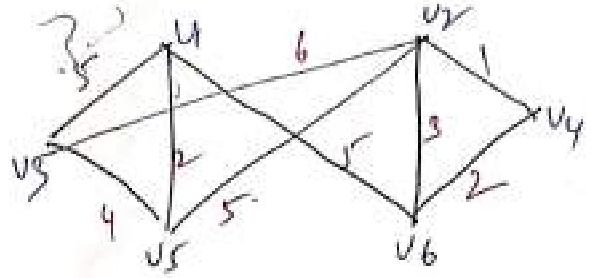


the minimum cost is $= 3+2+1+2$

$= \boxed{8 \text{ units.}}$

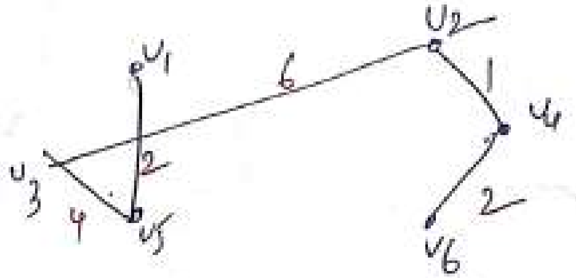
4.31

Find MST for the following graph



Sol: Selected edges are

(V1, V5), (V5, V3), (V3, V2), (V2, V4) (V4, V6).

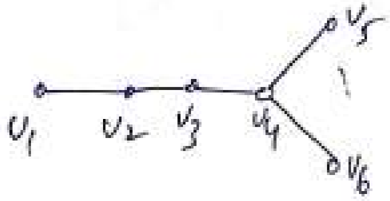


minimum cost : 2+4+6+1+2

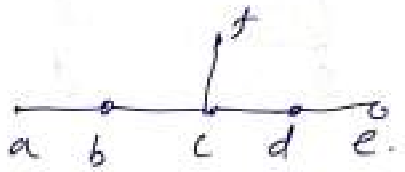
$\Rightarrow$ $\boxed{15 \text{ units}}$

Example Two Graphs are isomorphic or not.

G



G′



$V = 6$

$E = 5$

$V = 6$

$E = 5$

degree sequence

$\deg(v_1) = 1$
$(v_2) = 2$
$(v_3) = 2$
$(v_4) = 3$
$(v_5) = 1$
$(v_6) = 1$

degree sequence

$\deg(a) = 1$
$(b) = 2$
$(c) = 3$
$(d) = 2$
$(e) = 1$
$(f) = 1$

1 1 1 2 2 3

1 1 1 2 2 3

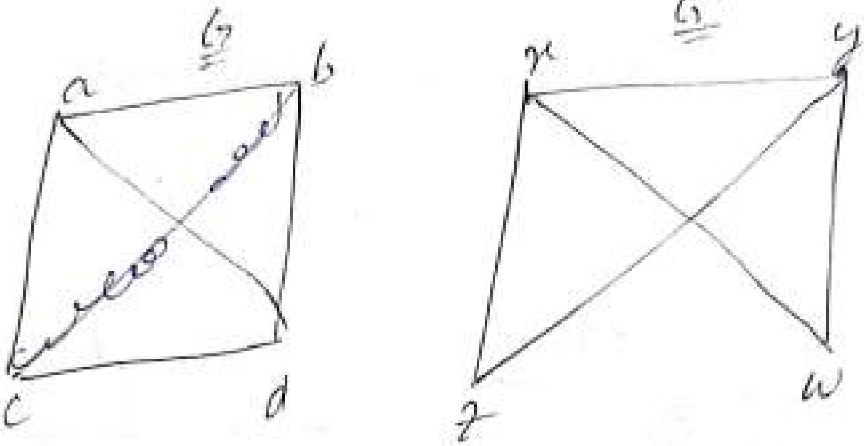
$f(v_1) = a,\ f(v_2) = b,\ f(v_3) = d,\ f(v_4) = d(b)\,c.$
$f(v_5) = d(e),\ f(v_6) = f.$

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	1	0	0	0	0
v_2	1	0	1	0	0	0
v_3	0	1	0	1	0	0
v_4	0	0	1	0	1	1
v_5	0	0	0	1	0	0
v_6	0	0	0	1	0	0

	a	b	d	c	e	f
a v_1	0	1	0	0	0	0
b v_2	1	0	0	1	0	0
d v_3	0	0	0	1	1	0
c v_4	0	1	1	0	0	1
e v_5	0	0	1	0	0	0
f v_6	0	0	0	1	0	0

not isomorphic

Ex:-



Sol:- 1) no: of vertices — 4 = 4

2) no: of Edges — 5 = 5.

3) Degree Sequence — a – 3 x – 3

b – 2 y – 3

c – 2 z – 2

d – 3 w – 2

G – 2, 2, 3, 3 G' – 2, 2, 3, 3.

4) Mapping

a – 3 < 2, 2, 3

b – 2 < 3, 3

c – 2 < 3, 3

d – 3 < 2, 2, 3

$f(a) = x$. x

$f(b) = z$ y

$f(c) = w$ z

$f(d) = y$. w

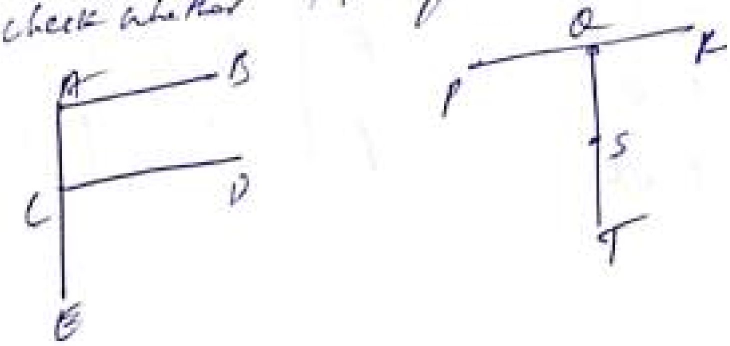
	a	b	c	d
a	0	1	1	1
b	1	0	0	1
c	1	0	0	1
d	1	1	1	0

	x	z	w	y
x	0	1	1	1
z	1	0	0	1
w	1	0	0	1
y	1	1	1	0

Both adjacency matrix are same.

$\therefore$ two graphs are isomorphic.

Ex: check whether following graphs are isomorphic or not.



1) no. of vertices — 5   5

2) edges — 4   4

3) degree sequence

(A, B, C, D, E) — (2, 1, 3, 1, 1)

3 2 1 1 1

degree sequence

(P Q R S T) — (1, 3, 1, 2, 1).

3 2 1 1 1

4) C (3) — A - 2, D - 1, E - 1

A - S
D - P
E - R.

Q (3) — P - 1, R - 1, S - 2

A - S
B - T
C - Q
D - R
E - P.

A (2) — B - 1, C 3

S (2) — Q - 3, T - 1

G G'

5) Edge preserving property

A-B ⇒ S-T
A-C ⇒ S-Q
C-D ⇒ Q-R
C-E ⇒ Q-P

P-Q ⇒ E-C
Q-R ⇒ C-D
Q-S ⇒ C-A
S-T ⇒ A-B

6) Adjacency matrix

	A	B	C	D	E
A	0	1	1	0	0
B	1	0	0	0	0
C	1	0	0	1	1
D	0	0	1	0	0
E	0	0	1	0	0

	S	T	Q	R	P
S	0	1	1	0	0
T	1	0	0	0	0
Q	1	0	0	1	1
R	0	0	1	0	0
P	0	0	1	0	0

4.34

Binary Operations (or) operations on Relations

The following operations are performed on the relations

① union
② Intersection
③ set Difference
④ complement
⑤ Inverse operation (or) converse.

① – ③ } Binary operations

④ – ⑤ } unary operations.

1. Union:

If R and S are two relations from set A to set B. then the union of R and S relations are denoted by $R \cup S$.

It can be defined as.

$$R \cup S = \{(a,b) / a \in A, b \in B, (a,b) \in R \text{ (or) } (a,b) \in S\}.$$

2. Intersection:

If R and S are two relations from set A to set B. the intersection of two relations R and S is denoted by $R \cap S$.

It can be defined as.

$$R \cap S = \{(a,b) / a \in A, b \in B, (a,b) \in R \text{ and } (a,b) \in S\}$$

3. Set Difference:

If R and S are two relations from set A to set B., then set Difference of two relations R and S is denoted by $R - S$, It can be defined as.

$R-S = \{(a,b) / a \in A, b \in B, (a,b) \in R$ and $(a,b) \notin S\}$.

$S-R = \{(a,b) / a \in A, b \in B, (a,b) \in S$ and $(a,b) \notin R\}$.

$R-S \neq S-R$.

4. Complement operation:

let R be a relation, then the complement of relation 'R' is denoted by R^c or $\bar{R}$ or R'.

It can be defined as.

$R^c = \{(a,b) / (a,b) \in A \times B$ and $(a,b) \notin R\}$.

$R^c = (A \times B) - R$.

$S^c = \{(a,b) / (a,b) \in A \times B$ and $(a,b) \notin S\}$.

$S^c = (A \times B) - S$

5- Inverse operations

let 'R' be a relation, the inverse operation on a relation R is denoted by R^{-1}.

It can be defined as.

$R^{-1} = \{(a,b) / (b,a) \in R^{-1}, a \in A, b \in B\}$.

$S^{-1} = \{(a,b) / (b,a) \in S^{-1}, a \in A, b \in B\}$.

Ex: $(a,b) \in R$

$(b,a) \in R^{-1}$.

Examples.

Example
① let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. Relation 'R' defined on set A and relation 'S' defined on set B.

$R = \{(1,1),(2,2),(3,3)\}$. and $S = \{(1,1),(1,2),(1,3),(1,4)\}$.

Find out

1) $R \cup S$ 3) $R-S$ 5) R^c ⑦ R^{-1}

2) $R \cap S$ 4) $S-R$ 6) S^c 8) S^{-1}.

Solution let $A = \{1,2,3\}$, $B = \{1,2,3,4\}$.

$$R = \{(1,1),(2,2),(3,3)\}$$

$$S = \{(1,1),(1,2),(1,3),(1,4)\}$$

① $R \cup S = \{(1,1),(2,2),(3,3),(1,2),(1,3),(1,4)\}$.

② $R \cap S = \{(1,1)\}$

③ $R-S = \{(2,2),(3,3)\}$

④ $S-R = \{(1,2),(1,3),(1,4)\}$

⑤ $R^c = \{(1,2),(1,3),(2,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4)\}$ $R^c = (A \times B) - R$.

⑥ $S^c = \{(2,2),(3,3),(4,4),(2,1),(3,1),(3,2),(2,3),(2,4),(3,4)\} \Rightarrow S^c = (A \times B) - S$.

⑦ $R^{-1} = \{(1,1),(2,2),(3,3)\}$

⑧ $S^{-1} = \{(1,1),(2,1),(3,1),(4,1)\}$.

Ex: let $A = \{1,2\}$, $B = \{1,2,4,5\}$, $C = \{5,7,9,10\}$.

find ① $A \cup B$
2) $A \cap B$
3) $(A \cup B) \cap C$
4) $(A \cap B) \cap C$.

Ex: let $R = \{(1,2), (3,4), (2,2)\}$ and $S = \{(4,2), (2,5), (3,1), (1,3)\}$ be relations from A to B. where
$A = \{1,2,3,4\}$ $B = \{1,2,3,4,5\}$.
find $R \circ S$, $S \circ R$, $R \circ R$, $S \circ S$, $R \circ R \circ R$.

Representation of relations

A relation can be represented in 2 ways.

1. Relation matrix
2. Digraph of a relation

Relation Matrix:

let $A = \{a_1, a_2, \ldots a_m\}$, and $B = \{b_1, b_2, \ldots b_n\}$ are finite sets containing m, n elements. and R is relation from set A to set B.
we can represent 'R' by $m \times n$ (read it as m by n) matrix called "relation matrix" denoted by M_R.

$$M_R = [M_{ij}]$$

$$\text{where } M_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

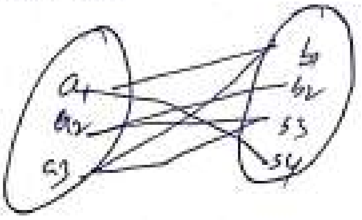
Ex: If $A = \{a_1, a_2, a_3\}$, and $B = \{b_1, b_2, b_3, b_4\}$ then the relation R from A to B is given by.

$$R = \{(a_1,b_1), (a_1,b_4), (a_2,b_2), (a_2,b_3), (a_3,b_1), (a_3,b_3)\}.$$

Sol: $A = \{a_1, a_2, a_3\}$ 3 elements

$B = \{b_1, b_2, b_3, b_4\}$. 4 elements.

$R: A \to B$.



$M_R = M_{ij}$

$$M_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

$$M_{ij} = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

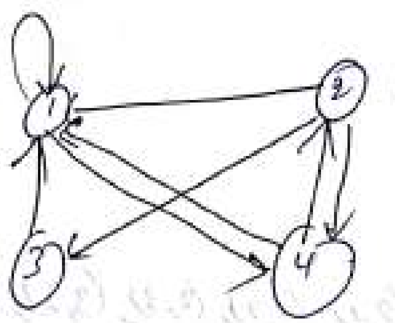
2- Digraph of a Relation:

A relation can be represented pictorially by drawing its digraph as follows.

1.

✓ let $A = \{1,2,3,4\}$ and R be a relation defined on set A as $R = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1)\}$. Draw the digraph of relation.

Sol: $A = \{1,2,3,4\}$.



Digraph of relation 'R'

$$R = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1)\}.$$

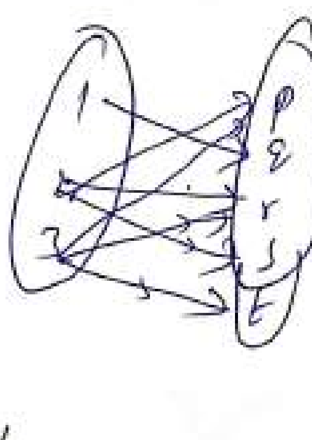
Ex: let $A = \{1,2,3\}$ and $B = \{p,q,r,s,t\}$ are two finite sets and R be a relation from set A to set B which is given by.

$$R = \{(1,q),(2,p),(2,r),(2,s),(3,p),(3,r),(3,t)\}.$$

Sol: $A = \{1,2,3\}$ $B = \{p,q,r,s,t\}$.

$$R = \{(1,q),(2,p),(2,r),(2,s),(3,p),(3,r),(3,t)\}.$$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} p & q & r & s & t \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$



$$M_R = m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & (a_i, b_j) \notin R \end{cases}$$

Example: let $A = \{1,2,3,4\}$ and let R be a relation defined on set A as.

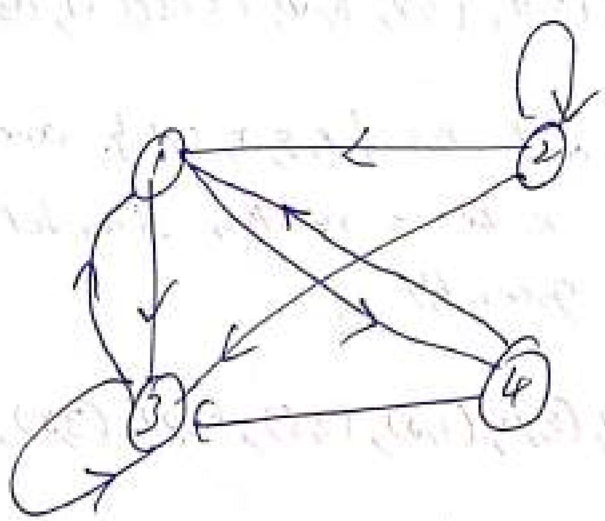
$R = \{(1,3), (1,4), (2,1), (2,2), (2,3), (3,3), (4,3), (3,1), (4,1)\}$.

Draw the digraph of relation R.

Sol: $A = \{1,2,3,4\}$.

$R: A \to A$.

$R = \{(1,3), (1,4), (2,1), (2,2), (2,3), (3,3), (4,3), (3,1), (4,1)\}$.



..7.

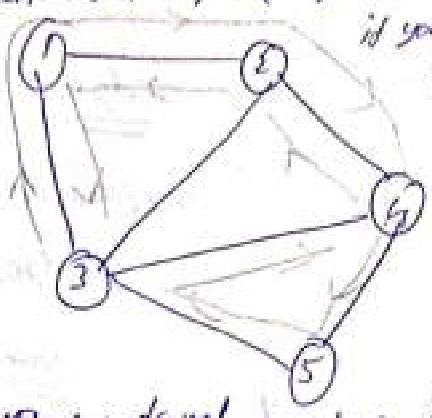
4:38

Walk, Trail, Circuit, Path, Cycle:

Starting and ending ~~vert~~ vertices are different.

Walk: A walk is a sequence of vertices and edges of a graph.

if you traverse a graph, then we get a walk.



Vertex } repeated
edge }

walk can be open or closed.

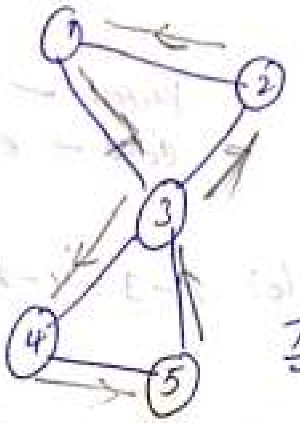
open:

closed:

In walk, you can travel anywhere i.e vertices and edges are repeated.
↓ what ever you wish.

Walk: 1-2-4-5-3-4-2-1-3.

Trail: Trail is a open walk in which no edge is repeated and vertices are repeated



Vertex — repeated

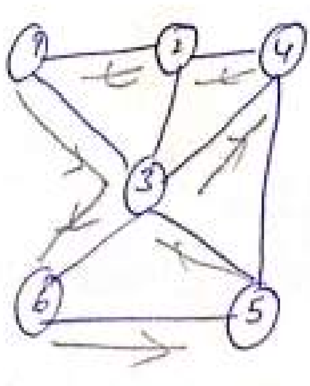
edges — should not repeated.

Trail: 1-3-4-5-3-2

Circuit:

circuit can be trail, if it is closed.

Trail should be closed.



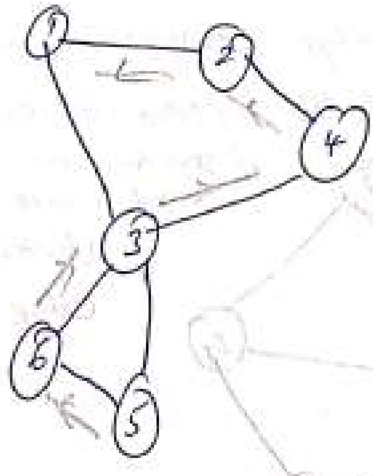
Vertex — repeated.

edge — should not be repeated.

should be starting and ending vertex same.

Circuit: 1-3-6-5-3-4-2-1.

Path :- It is trail in which neither vertex nor edges are repeated. starting and end vertex should be different.



vertex: not repeated

edge: not repeated

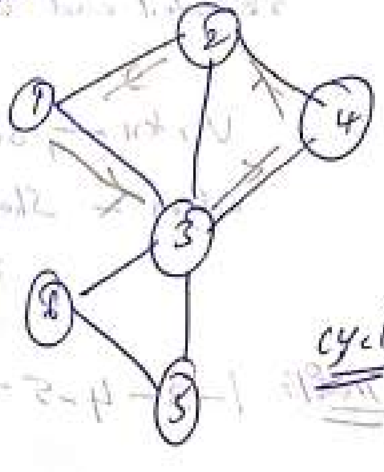
neither or not vertices and edges can be repeated.

Note: There can be multiple paths in the graph.

Path: start - 5, end - 1

Path: 5 - 6 - 3 - 4 - 2 - 1.

cycle: cycle is a path that can be closed.



vertex — repeated only starting and ending.

edge — edge not repeated

cycle: 1 - 3 - 4 - 2 - 1.

Note: Trail, circuit, path, cycle are walks in a graph.

Path matrix (Reachability matrix).

Using Warshall's algorithm

Given the adjacency matrix A of a simple di-graph, then the following steps produce the path matrix P (or A^+)

Step1: $P^{[0]} = A$

Step2: $k = 1$

Step3: $i = 1$

Step4: $P_{ij}^{[k]} = P_{ij}^{[k-1]} \vee \left[P_{ik}^{(k-1)} \wedge P_{kj}^{[k-1]} \right] \forall j = 1$ to n.

Step5: $i = i+1$, If $i \le n$, go to step4.

Step6: $k = k+1$, if $k \le n$ goto step3, otherwise stop.

Example: Calculate the path matrix P from the adjacency matrix (by Warshall's algorithm):

Sol: Given adjacency matrix $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Initially $P^{[0]} = A^0$

$$P^{[0]} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = A^0$$

Formula

$$P^{k}_{ij} = P^{k-1}_{ij} \vee \left[P^{k-1}_{ik} \wedge P^{k-1}_{kj} \right].$$

$$P^{[0]} = \begin{array}{c|cccc} & 11 & 12 & 13 & 14 \\ \hline & \left[\begin{matrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix}\right] \end{array}$$

initially $k=1$, $i=1$.

$i=1$ $j=1$
$$P^{1}_{11} = P^{0}_{11} \vee \left[P^{0}_{11} \wedge P^{0}_{11} \right]$$
$$= 0 \vee [0 \wedge 0] = 0 \vee 0 = 0.$$

$j=2$.
$$P^{1}_{12} = P^{0}_{12} \vee \left[P^{0}_{12} \wedge P^{0}_{12} \right]$$
$$= 1 \vee [0 \wedge 1]$$
$$= 1 \vee 0 = 1.$$

$j=3$
$$P^{1}_{13} = P^{0}_{13} \vee \left[P^{0}_{11} \wedge P^{0}_{13} \right]$$
$$= 0 \vee [0 \wedge 0] = 0 \vee 0 = 0.$$

$j=4$.
$$P^{1}_{14} = P^{0}_{14} \vee \left[P^{0}_{11} \wedge P^{0}_{14} \right]$$
$$= 1 \vee [0 \wedge 1]$$
$$= 1 \vee 0 = 1.$$

$k=1$. 8, 11 4-39

$i=2$

$j=1$, $P_{21}^{[1]} = P_{21}^{0} \vee [P_{21}^{0} \wedge P_{11}]$

$= 0 \vee [0 \wedge 0] = 0 \vee 0$

$j=2$ $P_{22}^{[1]} = P_{22}^{[0]} \vee [P_{21}^{[0]} \wedge P_{12}^{[0]}]$

$= 0 \vee [0 \wedge 1] = 0$.

$j=3$ $P_{23}^{[1]} = P_{23}^{0} \vee [P_{21}^{0} \wedge P_{13}^{0}]$

$= 0 \vee [0 \wedge 1] = 0$.

$j=4$ $P_{24}^{[1]} = P_{24}^{0} \vee [P_{21}^{0} \wedge P_{14}^{0}]$

$= 1 \vee [0 \wedge 1] = 1 \vee 0 = 1$.

$i=3$, $k=1$.

$j=1$, $P_{31}^{[1]} = P_{31}^{0} \vee [P_{31}^{0} \wedge P_{11}^{0}]$

$= 0 \vee (0 \wedge 0) = 0$.

$i=3$, $k=1$

$j=2$. $P_{32}^{[1]} = P_{32}^{0} \vee [P_{31}^{0} \wedge P_{12}^{0}]$

$= 1 \vee [0 \wedge 1] = 1$

$j=3$ $P_{33}^{1} = P_{33}^{0} \vee [P_{31}^{0} \wedge P_{13}^{0}]$

$= 0 \vee (0 \wedge 0) = 0$.

$j=4$ $P_{34}^{[1]} = P_{34}^{0} \vee [P_{21}^{0} \wedge P_{14}^{0}]$

$= 1 \vee [0 \wedge 1] = 1$

$i=4, k=1$

$j=1 \quad P_{41}^{[1]} = P_{41}^{0} \vee [P_{41}^{0} \wedge P_{11}]$

$= 0 \vee (0 \wedge 0) = 0.$

$j=2 \quad P_{42}^{[1]} = P_{42}^{0} \vee (P_{41}^{0} \wedge P_{12})$

$= 0 \vee (0 \wedge 1) = 0 \vee 0 = 0$

$j=3. \quad P_{43}^{[1]} = P_{43}^{0} \vee (P_{41}^{0} \wedge P_{13}^{0})$

$= 1 \vee (0 \wedge 1) = 1.$

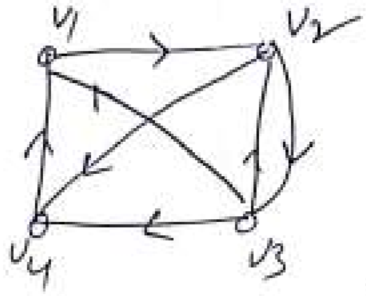
$j=4. \quad P_{44}^{[1]} = P_{44}^{0} \vee (P_{41}^{0} \wedge P_{14}^{0})$

$= 0 \vee (0 \wedge 1) = 0.$

$\boxed{k=2, i=1}$

$j=1. \quad P_{11}^{[2]} = P_{11}^{1} \vee [P_{12}^{1} \wedge P_{21}^{1}]$

Ex: Find the path matrix for the following graphs.



Sol: Adjacency matrix

$$A = \begin{array}{c|cccc} i \backslash j & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & \overset{11}{0} & \overset{12}{1} & \overset{13}{0} & \overset{14}{0} \\ v_2 & \overset{21}{0} & \overset{22}{0} & \overset{23}{1} & \overset{24}{1} \\ v_3 & \overset{31}{1} & \overset{32}{1} & \overset{33}{0} & \overset{34}{1} \\ v_4 & 1 & 0 & 0 & 0. \end{array}$$

$P^0 = A$.

Initially $i=1, k=1$.

$j=1 \quad P^1_{11} = P^0_{11} \vee (P^0_{11} \wedge P^0_{11}) = 0 \vee (0 \wedge 0) = 0.$

$j=2 \quad P^1_{12} = P^0_{12} \vee (P^0_{11} \wedge P^0_{12}) = 1 \vee (0 \wedge 1) = 1 \vee 0 = 1.$

$j=3 \quad P^1_{13} = P^0_{13} \vee (P^0_{11} \wedge P^0_{13}) = 0 \vee (0 \wedge 0) = 0 \vee 0 = 0$

$j=4 \quad P^1_{14} = P^0_{14} \vee (P^0_{11} \wedge P^0_{14}) = 0 \vee (0 \wedge 0) = 0 \vee 0 = 0.$

$i=2, k=1$

$j=1 \quad P^1_{21} = P^0_{21} \vee (P^0_{21} \vee P^0_{11}) = 0 \vee (0 \wedge 0) = 0.$

$j=2 \quad P^1_{22} = P^0_{22} \vee (P^0_{21} \vee P^0_{12}) = 0 \vee (0 \wedge 1) = 0 \vee 0 = 0.$

$j=3 \quad P^1_{23} = P^0_{23} \vee (P^0_{21} \vee P^0_{13}) = 1 \vee (0 \wedge 0) = 1 \vee 0 = 1$

$j=4 \quad P^1_{24} = P^0_{24} \vee (P^0_{21} \vee P^0_{14}) = 1 \vee (0 \wedge 0) = 1 \vee 0 = 1$

$$P^{(0)} = A = \begin{array}{c} \begin{array}{cccc} 11 & 12 & 13 & 14 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{4\times4} \end{array}$$

4

path matrix

$$P_{ij}^{k} = P_{ij}^{k-1} \vee \left[P_{ik}^{k-1} \wedge P_{kj}^{k-1} \right]$$

k = n - order vertices.

$$P^{[1]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & & & \\ & & & \end{bmatrix}$$

$k=1$

$i=1$

$j=1$ $\quad P_{11}^{1} = P_{11}^{0} \vee \left[P_{11}^{0} \wedge P_{11}^{0} \right]$

$= 0 \vee [0 \wedge 0] = 0 \vee 0 = 0$

$$P^{4} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$j=2$ $\quad P_{12}^{1} = P_{12}^{0} \vee [P_{11}^{0} \wedge P_{12}^{0}] = 1 \vee [0 \wedge 1] = 1 \vee 0 = 1$

$j=3$ $\quad P_{13}^{1} = P_{13}^{0} \vee [P_{11}^{0} \wedge P_{13}^{0}] = 0 \vee [0 \wedge 0] = 0 \vee 0 = 0$

$j=4$ $\quad P_{14}^{1} = P$

$k=1$ $\quad$ $k=1$ $\quad$ $k=1$

$i=2$ $\quad$ $i=3$ $\quad$ $i=4$

$j=1$

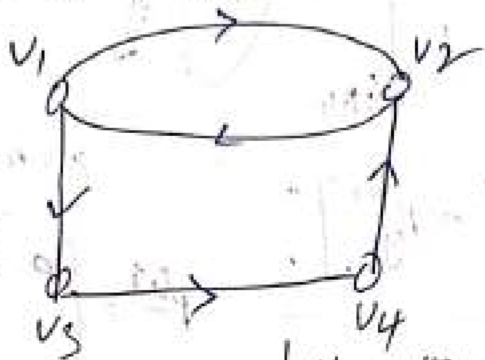
$j=2$

$j=3$

$j=4.$

Ex: Consider the following digraph, use its adjacency Matrix to find how many paths of length 3 exists from v_1 to v_2.

Sol:



Sol: $A(G) =$

$$A(0) \quad \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ & 11 & 12 & 13 & 14 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_2 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 1 \\ v_4 & 0 & 1 & 0 & 0 \end{array} = P(0).$$

Initially $i=1, k=1$.

$j=1.$ $\quad P^1_{11} = P^0_{11} \vee (P^0_{11} \wedge P^0_{11}) = 0 \vee (0 \wedge 0) = 0 \vee 0 = 0$

$j=2$ $\quad P^1_{12} = P^0_{12} \vee (P^0_{11} \wedge P^0_{12}) = 1 \vee (0 \wedge 1) = 1 \vee 0 = 1$

$j=3$ $\quad P^1_{13} = P^0_{13} \vee (P^0_{11} \wedge P^0_{13}) = 1 \vee (0 \wedge 1) = 1 \vee 0 = 1.$

$j=4$ $\quad P^1_{14} = P^0_{14} \vee (P^0_{11} \wedge P^0_{14}) = 0 \vee (0 \wedge 0) = 0 \vee 0 = 0.$

$i=2, k=1$

$j=1$ $\quad P^1_{21} = P^0_{21} \vee (P^0_{21} \wedge P^0_{11}) = 0 \vee (0 \wedge 0) = 0.$

$j=2$ $\quad P^1_{22} = P^0_{22} \vee (P^0_{21} \wedge P^0_{12}) = 0 \vee (1 \wedge 1) = 0 \vee 1 = 1.$

$j=3$ $\quad P^1_{23} = P^0_{23} \vee (P^0_{21} \wedge P^0_{13}) = 0 \vee (1 \wedge 1) = 0 \vee 1 = 1.$

$j=4$ $\quad P^1_{24} = P^0_{24} \vee (P^0_{21} \wedge P^0_{14}) = 0 \vee (1 \wedge 0) = 0 \vee 0 = 0.$

$i=3,\ k=1$

$j=1 - p^0_{31} = p^0_{31} \vee (p^0_{31} \wedge p^0_{11}) = 0 \vee (0 \wedge 0) = 0$

$j=2 - p^0_{32} = p^0_{32} \vee (p^0_{31} \wedge p^0_{12}) = 0 \vee (0 \wedge 1) = 0$

$j=3 - p^0_{33} = p^0_{33} \vee (p^0_{31} \wedge p^0_{13}) = 0 \vee (0 \wedge 0) = 0$

$j=4 - p^0_{34} = p^0_{34} \vee (p^0_{31} \wedge p^0_{14}) = 1 \vee (0 \wedge 0) = 1.$

$i=4,\ k=1$

$j=1 - p^0_{41} = p^0_{41} \vee (p^0_{41} \wedge p^0_{11}) = 0 \vee (0 \wedge 0) = 0$

$j=2 - p^0_{42} = p^0_{42} \vee (p^0_{42} \wedge p^0_{12}) = 1 \vee (1 \wedge 1) = 1$

$j=3 - p^0_{43} = p^0_{43} \vee (p^0_{43} \wedge p^0_{13}) = 0 \vee (0 \wedge 1) = 0$

$j=4 - p^0_{44} = p^0_{44} \vee (p^0_{44} \wedge p^0_{14}) = 0 \vee (0 \wedge 0) = 0$

$$A' = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_2 & 0 & 1 & 1 & 0 \\ v_3 & 0 & 0 & 0 & 1 \\ v_4 & 0 & 1 & 0 & 0 \end{array}$$

$i=1,\ k=2.$

$j=1,\quad P^2_{11} = P^1_{11} \vee (P^1_{12} \wedge P^1_{21}) = 0\vee(1\wedge 0)=0$

$j=2\quad P^2_{12} = P^1_{12} \vee (P^1_{12} \wedge P^1_{22}) = 1\vee(1\wedge 1)=1$

$j=3\quad P^2_{13} = P^1_{13} \vee (P^1_{12} \wedge P^1_{23}) = 1\vee(1\wedge 1)=1$

$j=4\quad P^2_{14} = P^2_{14} \vee (P^1_{12} \wedge P^1_{24}) = 0\vee(1\wedge 0)=0$

$i=2,\ k=2$

$j=1\quad P^2_{21} = P^1_{21} \vee (P^1_{22} \wedge P^1_{21}) = 0\vee(1\wedge 0)=0$

$j=2\quad P^2_{22} = P^1_{22} \vee (P^1_{22} \wedge P^1_{22}) = 1\vee(1\wedge 1)=1.$

$j=3\quad P^2_{23} = P^1_{23} \vee (P_{22} \wedge P^1_{23}) = 1\vee(1\wedge 1)=1$

$j=4\quad P^2_{24} = P^1_{24} \vee (P_{22} \wedge P^1_{24}) = 0\vee(1\wedge 0)=0$

$i=3,\ k=2.$ $=0$

$j=1\quad P^2_{31} = P^1_{31} \vee (P^1_{32} \wedge P^1_{21}) = 0\vee(0\wedge 0)=0$ 1

$j=2\quad P^2_{32} = P^1_{32} \vee (P^1_{32} \wedge P^1_{22}) = 0\vee(0\wedge 1)=0$ 1.

$j=3\quad P^2_{33} = P^1_{33} \vee (P^1_{32} \wedge P^1_{23}) = 0\vee(0\wedge 1)=0$ 0.

$j=4\quad P^2_{34} = P^1_{34} \vee (P^1_{32} \wedge P^1_{24}) = 1\vee(0\wedge 0)=1.$

$i=4,\ k=2$

$j=1\quad P^2_{41} = P^1_{41} \vee (P^1_{42} \wedge P^1_{21}) = 0\vee(0\wedge 0)=0$

$j=2\quad P^2_{42} = P^1_{42} \vee (P_{42} \wedge P^1_{22}) = 1\vee(1\wedge 1)=1$

$j=3\quad P^2_{43} = P^1_{43} \vee (P^1_{42} \wedge P^1_{23}) = 0\vee(0\wedge 1)=0$).

$j=4\quad P^2_{44} = P^1_{44} \vee (P^1_{42} \wedge P^1_{24}) = 0\vee(0\wedge 0)=0$

4.44

$$A^2 = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_2 & 0 & 1 & 1 & 0 \\ v_3 & 0 & 0 & 0 & 1 \\ v_4 & 0 & 1 & 0 & 0 \end{array}$$

$i=1,\ k=3.$

$j=1 \quad P_{11}^3 = P_{11}^2 \vee (P_{13}^1 \wedge P_{31}^1) = 0 \vee (1 \wedge 0) = 0$

$j=2 \quad P_{12}^3 = P_{12}^2 \vee (P_{13}^1 \wedge P_{32}^1) = 1 \vee (1 \wedge 0) = 1$

$j=3 \quad P_{13}^3 = P_{13}^2 \vee (P_{13}^1 \wedge P_{33}^1) = 1 \vee (1 \wedge 0) = 1$

$j=4 \quad P_{14}^3 = P_{14}^2 \vee (P_{13}^1 \wedge P_{34}^1) = 0 \vee (1 \wedge 1) = 1$

$i=2,\ k=3.$

$j=1 \quad P_{21}$

$j=2 \quad P_{22}$

$j=3 \quad P_{23}$

$j=4 \quad P_{24}$

· PTC

4.45

Relations:

set $A = \{1, 2\}$,
set $B = \{2, 3, 7\}$

Definition: let A & B are two sets. The cartesian product of A & B is defined as $A \times B$.

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

Cartesian product of 'n' sets $A_1, A_2 --- A_n$ is defined as $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2 \ldots a_n) \mid a_i \in A_i$ where $i = 1$ to $n\}$.

Example:- let $A = \{1,2,3,4\}$, $B = \{2,3,7\}$.

$A \times B = \{(1,2), (1,3), (1,7), (2,2), (2,3), (2,7), (3,2), (3,3), (3,7), (4,2), (4,3), (4,7)\}$.

$B \times A = \{(2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (7,1), (7,2), (7,3), (7,4)\}$.

Here $A \times B \neq B \times A$.

Note:- If A has m elements and B has 'n' elements then $A \times B$ & $B \times A$ will have "mn" elements.

If you are taking 3 sets

1) $(A \times B) \times C \neq A \times (B \times C)$.

2) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

$A \times (B \cap C) = (A \times B) \cap (A \times C)$.

~~Domain and Range of the~~

Binary Relation:

If A, and B are two sets, then a binary relation from set A to set B is a subset of A×B.

we say that x is related to y by R. which is written as xRy. if and only if $(x,y) \in R$.

denoted as $xRy \Leftrightarrow (x,y) \in R$.

⇒ if $A = B$ we say R is a (binary) relation on A.

we shall call a binary relation simply a relation.

Domain and Range:

let R be a relation from A to B. The Domain of R denoted by dom R is defined as.

$$\text{dom}R = \{x \mid x \in A \text{ and } (x,y) \in R \text{ for some } y \in B\}.$$

Ex: let $A = \{1,2\}$, $B = \{3,4\}$.

$$R = \{(1,3),(1,4),(2,3),(2,4)\}.$$

$$\text{dom}R = \{1,2\}.$$

Range of R, denoted by ran R is defined.

$$\text{ran}R = \{y \mid y \in B \text{ and } (x,y) \in R \text{ for some } x \in A\}.$$

$$\text{ran}R = \{3,4\}.$$

let $A = \{2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$.

Relation R is from A to B by $a, b \in R$.

$R = \{(2,4), (2,6), (3,3), (3,6), (4,4)\}$.

$\text{dom } R = \{2, 3, 4\}$

$\text{ran } R = \{3, 4, 6\}$.

Properties of the Relations

A relation 'R' on set A is said to be.

i) Reflexive:

if xRx or $(x,x) \in R \; \forall x \in A$.

ii) Irreflexive:

if $x \not R x$ or $(x,x) \notin R \; \forall x \in A$.

$xRy \Rightarrow yRx$

$xRy \Rightarrow y \not R x$

iii) Symmetric:-

if $xRy \Rightarrow yRx$ for all $x, y \in A$.

xRy

yRx

$x = y$.

iv) Anti Symmetric

if $xRy \Rightarrow yRx$ then $x = y$.

(OR.)

if $x \neq y$ and $xRy \Rightarrow y \not R x$ or $(y,x) \notin R$ for all $x, y \in A$.

v) transitive

if xRy and $yRz \Rightarrow xRz$. for all $x, y, z \in A$.

vi) assymmetric:-

if $xRy \Rightarrow y \not R x$ or $(y,x) \notin R$.

Ex: If $A = \{1,2,3,4\}$ then

$R = \{(1,1)(2,1)(1,3),(2,4)(3,3),(3,4)(4,4)\}$

is a relation reflexive?

Sol: The relation is reflexive as for every

$a \in A, (a,a) \in R$. i.e.

$\{(1,1),(2,2),(3,3),(4,4)\} \in R$.

Ex: $A = \{1,2,3\}$.

$R = \{(1,2),(2,2),(3,1)(1,3)\}$ is the relation irreflexive

$a \in A, (a,a) \notin R$.

But have $(2,2) \in R$ so it is irreflexive.

Ex: $A = \{1,2,3\}$,

$R = \{(1,1),(2,2),(1,2),(2,1),(2,3)(3,2)\}$ is a relation

R symmetric or not.

if $(a,b) \in R \Rightarrow (b,a) \in R$.

The relation is symmetric as for every $(a,b) \in R$.

we have $(b,a) \in R$.

$(1,2), (2,1), (2,3), (3,2) \in R$.

Ex: If $(a,b) \in R$ and $(b,a) \in R$ then $a = b$.

$A = \{1,2,3\}$, $R = \{(1,1),(2,2)\}$. is the relation

R antisymmetric.

Sol R is antisymmetric $a = b$ when (a,b) &

$(b,a) \in R$.

$(1,1)$ & $(2,2)$ $\in R$.

Ex: let $A = \{4, 5, 6\}$

$R = \{(4,4), (4,5), (5,4), (5,6), (4,6)\}$

not Antisymmetric.

$4 \neq 5$, but $(4,5), (5,4) \in R$.

Ex: $A = \{1,2,3\}$, $R = \{(1,2), (2,1), (1,1), (2,2)\}$.

is a relation transitive.

Sol: It is transitive.

$(1,2), (2,1) \in R$ then $(1,1) \in R$.

~~Equivalen~~ Equivalence Relation

If a relation is reflexive, symmetric and transitive at the same time is known as equivalence relation.

Ex: Show that relation R is an equivalence relation in the set $A = \{1,2,3,4,5\}$ given by relation

$R = \{(a,b) : |a-b| \text{ is even}\}$.

Sol: $R = \{(a,b) : |a-b| \text{ is even}\}$ where $a,b \in A$.

1) Reflexive:

$a-a = |0| = 0$. 0 is always even.

$\therefore (a,a) \in R$.

Symmetric property

from the given relation.

$|a-b| = |b-a|$.

we know that $|a-b| = |-(b-a)| = |b-a|$.

$|a-b|$ is even.

$|b-a|$ also even.

$\therefore$ if $(a,b) \in R$ then $(b,a) \in R$.

$\therefore R$ is symmetric.

Transitive:

if $|a-b|$ is even. then $(a-b)$ is even. Similarly.

if $|b-c|$ is even the $(b-c)$ is even.

Sum of the even numbers also even.

So we can ~~write~~ write.

$a-b+b-c = a-c$ is also even.

so $|a-c|$ & $|b-c|$ is even then $|a-c|$ is also even.

$\therefore$ i.e if $(a,b), (b,c) \in R$ then $(a,c) \in R$.