per year, with the interest compounded annually. How much money will be in the account after 30 years?

10.8. A factory makes custom sports vehicles at an increasing rate. In the first month only one vehicle is made, in the second month two vehicles are made an so on, with n vehicles made in the n^{th} month.

(a) Set up a recurrence relation for the number of vehicles produced in the first n months by this factory.

(b) How many vehicles are produced in the first year?

(c) Find an explicit formula for the number of vehicles produced in the first n months by this factory.

10.1.2 Solution of Linear Homogeneous Recurrence Relations with Constant Coefficients

A recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \qquad (10.4)$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$, is called a linear homogeneous recurrence relation of degree k with constant coefficients.

Note 10.1 The recurrence relation given in Eq. (10.4) is linear, since each a_i has the power 1 and no terms of the type $a_i a_j$ occurred.

Note 10.2 The *degree* of the recurrence relation is k, since a_n is expressed in terms of the previous k terms of the sequence, i.e., degree is the difference between the greatest and lowest subscripts of the members of the sequence occurring in the recurrence relation.

Note 10.3 The coefficients of the terms of the sequence are all *constants*. They are not functions of n.

Note 10.4 If $F(n) = 0$, then the recurrence relation is said to be *homogeneous*; otherwise, it is said to be *non-homogeneous*.

The recurrence relation given in Eq. (10.4) is homogeneous.

Example 10.9 Provide some examples of linear homogeneous recurrence relation. Also, give their degrees.

Solution (i) The recurrence relation $S_n = (0.09)S_{n-1}$ is a linear homogeneous recurrence relation of degree 1.

(ii) The recurrence relation $F_n = F_{n-1} + F_{n-2}$ is a linear homogeneous recurrence relation of degree 2.

(iii) The recurrence relation $a_n = a_{n-4}$ is a linear homogeneous recurrence relation of degree 4. ■

Example 10.10 Determine whether the following recurrence relations are linear homogeneous recurrence relations with constant coefficients:

(i) $a_n = 2a_{n-4} + a_{n-3}^2$

(ii) $H_n = 2H_{n-1} + 2$

(iii) $B_n = nB_{n-1}$

Solution (i) The recurrence relation $a_n = 2a_{n-4} + a_{n-3}^2$ is not linear.
(ii) The recurrence relation $H_n = 2H_{n-1} + 2$ is not homogeneous.
(iii) The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients. ■

Example 10.11 Determine which of the following recurrence relations are linear homogeneous recurrence relations with constant coefficients and also find their degrees:

(i) $a_n = 3a_{n-1} + 4a^2_{n-2} + 5a_{n-3}$
(ii) $a_n = 2na_{n-1} + a_{n-2}$
(iii) $a_n = a_{n-1} + a_{n-4}$
(iv) $a_n = a_{n-1} + 2$
(v) $a_n = a_{n-1}^2 + a_{n-2}$
(vi) $a_n = a_{n-2}$
(vii) $a_n = a_{n-1} + n$

Solution (i) This is a linear homogeneous recurrence relation with constant coefficients.

$$\text{Degree} = (n-3) - n = 3.$$

(ii) This does not have constant coefficients.
(iii) This is a linear homogeneous recurrence relation with constant coefficients.

$$\text{Degree} = (n-4) - n = 4.$$

(iv) This is not a homogeneous recurrence relation.
(v) This is not a linear recurrence relation.
(vi) This is a linear homogeneous recurrence relation with constant coefficients.

$$\text{Degree} = 2.$$

(vii) This is not a homogeneous recurrence relation. ■

Characteristic roots Consider the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$.

The characteristic equation of the recurrence relation given above is

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k \neq 0$$

The solutions of the characteristic equation are called the *characteristic roots*.

Theorem 10.1 Let c_1 and c_2 be real numbers. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof Let r_1 and r_2 be two distinct roots of the characteristic equation $r^2 - c_1 r - c_2 = 0$.

Let α_1, α_2 be two constants such that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$. (10.5)

We need to prove that $\{a_n\}$ is a solution of the recurrence relation.

Since r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$, we have

$$r_1^2 - c_1 r_1 - c_2 = 0 \Rightarrow r_1^2 = c_1 r_1 + c_2 \quad (10.6)$$

and $r_2^2 - c_1 r_2 - c_2 = 0 \Rightarrow r_2^2 = c_1 r_2 + c_2$ (10.7)

Now, $c_1 a_{n-1} + c_2 a_{n-2}$

$$= c_1 [\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}] + c_2 [\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}]$$

$= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2)$.

$= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2$ [from Eqs (10.6) and (10.7)]

$= \alpha_1 r_1^n + \alpha_2 r_2^n$

$= a_n$ [by our assumption]

$\therefore$ The sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation.

Conversely, we assume that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$ for some constants α_1, α_2 and $n = 0, 1, 2, \ldots$.

We need to prove that every solution $\{a_n\}$ of the recurrence relations has the form

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some constants α_1 and α_2 and $n = 0, 1, 2, \ldots$.

Suppose $\{a_n\}$ is a solution of the recurrence relation and the initial conditions $a_0 = c_0$ and $a_1 = c_1$ hold. We need to show that there are constants α_1 and α_2 so that the sequence $\{a_n\}$ with

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the initial conditions.

Now

$$a_0 = c_0 = \alpha_1 r_1^0 + \alpha_2 r_2^0 = \alpha_1 + \alpha_2$$

$$a_1 = c_1 = \alpha_1 r_1 + \alpha_2 r_2 \quad \text{[by our assumption]}$$

i.e., $c_0 = \alpha_1 + \alpha_2$

and $c_1 = \alpha_1 r_1 + \alpha_2 r_2$

Solving this, we get

$$\alpha_1 = \frac{c_1 - c_0 r_2}{r_1 - r_2}$$

and $\alpha_2 = \dfrac{c_0 r_1 - c_1}{r_1 - r_2}$

The values of α_1 and α_2 are valid only if $r_1 \neq r_2$.

Therefore, for the above values of α_1 and α_2, the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the two initial conditions.

Since $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ with $a_0 = c_0$ and $a_1 = c_1$ uniquely determine the sequence, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution form.

Theorem 10.2 Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof First, we show that if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, then the sequence $\{a_n\}$ is a solution of the recurrence relation. Since r_0 is a root of the characteristic equation $r^2 - c_1 r - c_2 = 0$

r_0 is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ [by Theorem 10.1] (10.8)

Now we need to prove that $a_n = n r_0^n$ is also a solution of Eq. (10.8).

Since r_0 is a root of $r^2 - c_1 r - c_2 = 0$ (10.9)

and the degree of equation (10.9) is 2

$$r^2 - c_1 r - c_2 = (r - r_0)^2$$
$$= r^2 - 2r_0 r + r_0^2$$

Equating the corresponding coefficients, we have

$$c_1 = 2r_0,\ c_2 = -r_0^2$$

Now

$$c_1 a_{n-1} + c_2 a_{n-2} = c_1 [(n-1)r_0^{n-1}] + c_2 [(n-2)r_0^{n-2}]$$
$$= 2r_0 (n-1)r_0^{n-1} - r_0^2 (n-2)r_0^{n-2}$$
$$= r_0^n [2(n-1) - (n-2)]$$
$$= nr_0^n$$
$$= a_n$$

$\therefore$ nr_0^n is a solution.

Hence, by Theorem 10.7, $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ is a solution of Eq. (10.8).

Conversely, we have to prove that every solution $\{a_n\}$ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for some constants α_1 and α_2 and for $n = 0, 1, 2, \ldots$.

Suppose that $\{a_n\}$ is a solution of the recurrence relation and the initial conditions $a_0 = c_0$ and $a_1 = c_1$ hold. We need to show that there are constants α_1 and α_2 so that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ satisfies the initial conditions.

$$a_0 = c_0 = \alpha_1$$
$$a_1 = c_1 = \alpha_1 r_0 + \alpha_2 r_0$$

i.e., $c_1 - \alpha_1 r_0 = \alpha_2 r_0$

i.e., $\alpha_2 = \left[\dfrac{c_1 - c_0 r_0}{r_0}\right]$

Therefore, when $\alpha_1 = c_0$, $\alpha_2 = \left[\dfrac{c_1 - c_0 r_0}{r_0}\right]$, the sequence $\{a_n\}$ with $\alpha_1 r_0^n + \alpha_2 n r_0^n$ satisfies the two initial conditions.

Since the recurrence relation and these initial conditions uniquely determine the sequence,

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

Theorem 10.3 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation. $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$ for $n = 0, 1, 2, \ldots$ where $a_1, a_2, \ldots, a_k$ are constants.

Theorem 10.4 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively, so that $m_i \geq 1$ *for* $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1})$$
$$r_2^n + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{mt-1}) r_t^n,$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \le i \le t$.

Note 10.5 Consider the recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, $n \ge 2$.

The characteristic equation is $r^2 - c_1 r - c_2 = 0$

Let the roots of the characteristic equation be r_1 and r_2.

Case (i) If r_1 and r_2 are real and distinct, then the solution is $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where α_1 and α_2 are arbitrary constants.

Case (ii) If r_1 and r_2 are real and equal, then the solution is $a_n = (\alpha_1 + \alpha_2 n) r^n$ where α_1 and α_2 are arbitrary constants.

Case (iii) If r_1 and r_2 are complex numbers, then the solution is $a_n = r^n (\alpha_1 \cos n\theta + \alpha_2 \cos n\theta$, where α_1 and α_2 are arbitrary constants.

Example 10.12 Solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}, \text{ for } n \ge 2, a_0 = 1, a_1 = 0.$$

Solution The given recurrence relation is $a_n - 5a_{n-1} + 6a_{n-2} = 0$.

The characteristic equation of the recurrence relation is $r^2 - 5r + 6 = 0$.

$\therefore r = 2, 3$

Hence, the solution is $a_n = c_1 2^n + c_2 3^n$, where c_1 and c_2 are constants. (10.10)

Initial conditions are $a_0 = 1, a_1 = 0$.

Now, $a_0 = 1 \Rightarrow c_1 2^0 + c_2 3^0 = 1$

$\Rightarrow c_1 + c_2 = 1$ (10.11)

$$a_0 = 0 \Rightarrow c_1 2^1 + c_2 3^1 = 0$$

$\Rightarrow 2c_1 + 3c_2 = 0$ (10.12)

Solving Eqs (10.11) and (10.12) we have, respectively,

$\Rightarrow c_1 = 1 - c_2$

$\Rightarrow 2(1 - c_2) + 3c_2 = 0$

$\Rightarrow c_2 = -2$

Also, $c_1 = 1 - (-2) = 3$

$\therefore c_1 = 3$ and $c_2 = -2$

Hence, the unique solution is

$$a_n = 3(2^n) - 2(3^n) \text{ for } n \ge 2$$

■

Example 10.13 Solve the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2} \text{ for } n \ge 2, a_0 = 16, a_1 = 80.$$

Solution The given recurrence relation is

$$a_n - 8a_{n-1} + 16a_{n-2} = 0$$

The characteristic equation is

$$r^2 - 8r + 16 = 0$$

$\Rightarrow (r-4)^2 = 0$

$\Rightarrow r = 4, 4$

Hence, the solution is

$$a_n = c_1 4^n + c_2 n 4^n, \text{ where } c_1 \text{ and } c_2 \text{ are arbitrary constants.} \qquad (10.13)$$

Initial conditions:

$$a_0 = 16 \Rightarrow c_1.4^0 + c_2.0 = 16$$

$\Rightarrow c_1 = 16$

Also, $a_0 = 80 \Rightarrow c_1 4 + c_2 1\,(4^1) = 80$

$\Rightarrow 4c_1 + 4c_2 = 80$

$\Rightarrow c_1 + c_2 = 20$

$\Rightarrow c_2 = 4$, since $c_1 = 16$

Hence, the unique solution is

$$a_n = 16\,(4^n) + 4n\,(4^n) = 4^{n+2} - n4^{n+1}$$

$\Rightarrow a_n = (4-n)\,4^{n+1}, n \geq 2$

Example 10.14 Find the solution of the recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$ and $a_2 = 0$.

Solution The given recurrence relation is

$$a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$$

The characteristic equation is

$$r^3 - 2r^2 - r + 2 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & -2 & -1 & 2 \\ & 0 & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$\Rightarrow (r-1)\,(r^2 - r - 2) = 0$

$\Rightarrow (r-1)\,(r+1)\,(r-2) = 0$

$\Rightarrow r = 1, 2, -1$

Hence, the solution

$$a_n = c_1 1^n + c_2 2^n + c_3 (-1)^n, \text{ where } c_1, c_2, c_3 \text{ are arbitrary constants.} \qquad (10.14)$$

Initial conditions are $a_0 = 3$ $a_1 = 6$ and $a_2 = 0$.

When $a_0 = 3$, $c_1 + c_2 + c_3 = 3$ (10.15)

When $a_1 = 6$, $c_1 + c_2 2^1 + c_3 (-1)^1 = 6$

$\Rightarrow c_2 + 2c_2 - c_3 = 6$ (10.16)

When $a_2 = 0$, $c_1 + c_2 2^2 + c_3 (-1)^2 = 0$

$\Rightarrow c_2 + 4c_2 + c_3 = 0$ (10.17)

Adding Eqs (10.15) and (10.16) $\Rightarrow 2c_1 + 3c_2 = 9$ (10.18)

Adding Eqs (10.16) and (10.17) $\Rightarrow 2c_1 + 6c_2 = 6$ (10.19)

Subtracting Eq. (10.19) from Eq. (10.20) $\Rightarrow -3c_2 = 3$

$\Rightarrow c_2 = -1$

Equation (10.18) $\Rightarrow 2c_1 = 9 - 3c_2$

$\Rightarrow 2c_1 = 9 + 3$

$\Rightarrow c_1 = 6$

Equation (10.15) $\Rightarrow c_3 = 3 - c_1 - c_2$

$= 3 - 6 + 1$

$\Rightarrow c_3 = -2$

$\therefore$ The unique solution is $a_n = 6(1^n) - 2^n - 2(-1)^n$.

Example 10.15 Find an explicit formula for the Fibonacci numbers.

Solution The Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, ... The recurrence relation corresponding to the Fibonacci sequence $\{F_n\}$, $n \geq 0$, is

$F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with the initial conditions $F_0 = 0$, $F_1 = 1$.

The characteristic equation of the recurrence relation is $r^2 - r - 1 = 0$.

Solving it, we have $r = \dfrac{1 \pm \sqrt{5}}{2}$

Hence, the solution of the recurrence relation is

$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{10.20}$$

where c_1 and c_2 are arbitrary constants.

Initial conditions are $F_0 = 0$ and $F_1 = 1$.

Now, $F_0 = 0 \Rightarrow c_1 + c_2 = 0$ (10.21)

And $F_1 = 1 \Rightarrow c_1\left(\dfrac{1+\sqrt{5}}{2}\right) + c_2\left(\dfrac{1-\sqrt{5}}{2}\right) = 1$ (10.22)

Solving Eqs (10.21) and (10.22):

Equation (10.21) $\Rightarrow c_1 = -c_2$.

$\therefore$ Equation (10.22) $\Rightarrow c_2\left[\left(\dfrac{1-\sqrt{5}}{2}\right) - \left(\dfrac{1+\sqrt{5}}{2}\right)\right] = 1$

$\Rightarrow c_2\left[-\sqrt{5}\right] = 1$

$\Rightarrow c_2 = \dfrac{-1}{\sqrt{5}}$

Using this in Eq. (10.21), we get

$$c_1 = \frac{1}{\sqrt{5}}$$

The solution is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, n \geq 0$$

■

Example 10.16 The *Lucas numbers* satisfy the recurrence relation $L_n = L_{n-1} + L_{n-2}$ and the initial conditions $L_0 = 2$ and $L_1 = 1$.

(i) Show that $L_n = F_{n-1} + F_{n-2}$ for $n = 2, 3, \ldots$, where F_n is the nth Fibonacci number.
(ii) Find an explicit formula for the Lucas numbers.

Solution (i) Let S_n be the statement

$$L_n = F_{n-1} + F_{n+1} \text{ for } n = 2, 3, \ldots.$$

We shall prove this by using the principle of mathematical induction.

Basic step: S_2 is shown to be true. That is, we need to prove

$$L_2 = F_1 + F_3.$$

By the definition of Lucas number, we can write

$$L_2 = L_1 + L_0$$
$$= 1 + 2, \text{ since } L_0 = 2, L_1 = 1$$
$$= 3$$

R.H.S. $S_2 = F_1 + F_3$

$= 1 + 2$, since from the Fibonacci number $0(F_0)\ 1(F_1)\ 1(F_2)\ 2(F_3)\ 3(F_4) \ldots$

$= 3$

$\Rightarrow L_2 = F_1 + F_3$

Hence, S_2 is true.

■

nductive step: We assume that S_k is true for every $k \leq n$.

$\Rightarrow L_k = F_{k-1} + F_{k+1}$ is true for every $k = n$.

We need to prove that S_{k+1} is true. That is, we need to prove

$L_{k+1} = F_k + F_{k+2}$.

Now, by the definition of Lucas numbers

$$L_{k+1} = L_k + L_{k-1}$$
$$= (F_{k-1} + F_{k+1}) + (F_{k-2} + F_k) \quad \text{[by our assumption]}$$
$$= (F_{k-1} + F_{k-2}) + (F_{k+1} + F_k)$$
$$= F_k + F_{k+2} \quad \text{[by the definition of Fibonacci numbers]}$$

i.e, S_{k+1} is true.

Hence, by the principle of mathematical induction,

S_n is true for every n.

(ii) The given recurrence relation is

$L_n - L_{n-1} - L_{n-2} = 0$

The corresponding characteristic equation is

$$r^2 - r - 1 = 0$$

$$\Rightarrow r = \frac{1 \pm \sqrt{5}}{2}$$

Let $r_1 = \dfrac{1+\sqrt{5}}{2}$ and $r_2 = \dfrac{1-\sqrt{5}}{2}$

Therefore, the solution is

$$L_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad (10.23)$$

where c_1 and c_2 are constants.

Initial conditions are $L_0 = 2$, $L_1 = 1$.

Now, $L_0 = 2 \Rightarrow c_1 + c_2 = 2$ (10.24)

$$L_1 = 1 \Rightarrow c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \quad (10.25)$$

Solving Eqs (10.24) and (10.25), we get

$$\alpha_1 = 1 \text{ and } \alpha_2 = 1$$

Hence, the unique solution is

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Example 10.17 Solve the recurrence relation

$$a_n = 2a_{n-1} - 2a_{n-2},\ a_0 = 1,\ a_1 = 2.$$

Solution The given recurrence relation is $a_n - 2a_{n-1} + 2a_{n-2} = 0$.

Its characteristic equation is $r^2 - 2r + 2 = 0$

$$\Rightarrow r = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= 1 \pm i$$

The modulus–amplitude form of

$$1 \pm i = \sqrt{2}\left(\cos\frac{\pi}{4} \pm \sin\frac{\pi}{4}\right)$$

$\therefore$ The general solution of the recurrence relation is

$$a_n = \left(\sqrt{2}\right)^n \left(c_1 \cos\frac{n\pi}{4} \pm c_2 \sin\frac{n\pi}{4}\right) \tag{10.26}$$

where c_1 and c_2 are constants.

Given that $a_0 = 1$ and $a_1 = 2$.

Now, $a_0 = 1 \Rightarrow c_1 \pm 0 = 1$

$\Rightarrow c_1 = 1.$

Also, $a_1 = 2 \Rightarrow \left(\sqrt{2}\right)\left(c_1 \cos\frac{\pi}{4} + c_2 \sin\frac{\pi}{4}\right) = 2$

$\Rightarrow \sqrt{2}\left[c_1 \frac{1}{\sqrt{2}} + c_2 \frac{1}{\sqrt{2}}\right] = 2$

$\Rightarrow c_1 + c_2 = 2$

$\Rightarrow c_2 = 1.$

$\therefore$ The required solution is

$$a_n = \left(\sqrt{2}\right)\left(\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right)$$

■

10.1.3 Solution of Non-homogeneous Recurrence Relation or Inhomogeneous Recurrence Relation

A linear inhomogeneous or non-homogeneous recurrence relation with constant coefficients of degree k is a recurrence relation of the form

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + G(n)$, where $c_1, c_2, \ldots, c_k$ are real numbers and $G(n)$ is a function not identically zero depending only on n.

Algorithm for solving non-homogeneous finite-order linear recurrence relation To solve the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = G(n)$$

or

$$S(k) = c_1 S(k-1) + c_2 S(k-2) + \cdots + c_n S(k-n) = g(k),$$

we have to adopt the following procedure.

Step 1. We obtain the homogeneous solution.

First, we write the associated homogeneous recurrence relation, namely $G(n) = 0$

i.e., $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_n a_{n-k} = 0$

Then, we find its general solution, which is called the homogeneous solution.

Step 2. We obtain the particular solution.

There is no general procedure for finding the particular solution of a recurrence relation.

However, if $G(n)$ has any one of the following forms

(i) polynomial in n
(ii) a constant
(iii) powers of constant

then we may guess the forms of the particular solution and exactly find out by the method of undetermined coefficients.

Particular solution for given G(n) Table 10.2 shows the particular solution for given $G(n)$.

Table 10.2 Particular solution for given $G(n)$

S. No.	$G(n)$	*Form of particular solution*
(i)	A constant, c	A constant, d
(ii)	A linear function $c_0 + c_1 n$	A linear function $d_0 + d_1 k$
	N	$d_0 + d_1 k$
(iii)	n^2	$d_0 + d_1 k + d_2 k^2$
(iv)	An m[th] degree polynomial	An m^{th} degree polynomial
	$c_0 + c_1 n + c_2 n^2 + \cdots + c_m n^m$	$d_0 + d_1 k + d_2 k^2 + \cdots + d_m k^m$
(v)	$r^n, r \in R$	dr^n

Step 3. We substitute the guess from Step 2 into the recurrence relation. If the guess is correct, then we can determine the unknown coefficient of the guess. If we are not able to determine the constants, then our guess is wrong and hence we go to Step 2.

Step 4. The general solution of the recurrence relation is the sum of the homogeneous and particular solutions.

Step 10. If no initial conditions are given, then Step 4 will give the solution.

If n initial conditions are given, then we get n equations with n unknowns. Solving the system, we get a complete solution.

Example 10.18 Solve the recurrence relation $a_n = 3a_{n-1} + 2^n$, $a_0 = 1$

Solution The inhomogeneous recurrence relation is

$$a_n - 3a_{n-1} = 2^n \tag{10.27}$$

(i) The associated homogeneous equation is

$$a_n - 3a_{n-1} = 0$$

Its characteristic equation is

$$r - 3 = 0$$

$\Rightarrow r = 3$

$\therefore$ The homogeneous solution is

$$a_n(H) = c_1 3^n$$

(ii) Since the R.H.S. of the recurrence relation is 2^n and 2 is not the characteristic root, let the particular solution of the recurrence relation be

$$a_n = d2^n$$

Using this equation in the given recurrence relation, we get

$$d2^n - 3d2^{n-1} = 2^n$$

$\Rightarrow d - \frac{3}{2}d = 1$

$\Rightarrow 2d - 3d = 2$

$\Rightarrow d = -2$

$\therefore\ a_n^{(P)} = -2\,(2)^n = -2^{n+1}$

Hence, the general solution is

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$\Rightarrow a_n = c_1 3^n - 2^{n+1}$ (10.28)

Using the condition $a_0 = 1$, we get

$$a_0 = c_1 3^0 - 2^1 = 1$$

$\Rightarrow c_1 - 2 = 1$

$\Rightarrow c_1 = 3$

$\therefore$ The required solution is $a_n = 3\,(3^n) - 2^{n+1}$

i.e., $a_n = 3^{n+1} - 2^{n+1}$

Example 10.19 Solve the recurrence relation $a_n = 2a_{n-1} + 2^n$, $a_0 = 2$.

Solution The given recurrence relation is

$$a_n - 2a_{n-1} = 2^n \qquad (10.29)$$

(i) The associated homogeneous equation is $a_n - 2a_{n-1} = 0$.

The characteristic equation is

$$r - 2 = 0$$

$\Rightarrow r = 2$

$\therefore$ The homogeneous solution is $a_n^{(H)} = c_1 2^n$.

(ii) Since the R.H.S. of the recurrence relation is 2^n and 2 is the characteristic root, let $a_n = dn2^n$ be a particular solution of the recurrence relation.

Using this equation in the given recurrence relation, we get

$dn2^n - 2d(n-1)2^{n-1} = 2^n$

$\Rightarrow dn2^n - d(n-1)2^{n-1} = 2^n$

$\Rightarrow d\,[n - (n-1)] = 1$

$\Rightarrow d = 1$

$$a_n^{(P)} = n2^n$$

Hence, the general solution is

$$a_n = a_n^{(H)} + a_N^{(P)}$$

$\Rightarrow a_n = c_1 2^n + n2^n$ (10.30)

Given that $a_0 = 2$,

$\therefore c_1 2^0 + 0 = 2$

$\Rightarrow c = 2$

Therefore, the required solution is

$a_n\ 2(2^n) + n2^n$

$\Rightarrow a_n\ (2 + n) + n2^n$

■

Example 10.20 Solve the recurrence relation $a_n - 2a_{n-1} + a_{n-2} = 2$, with $a_0 = 25$, $a_1 = 16$.

Solution The given recurrence relation is

$$a_n - 2a_{n-1} + a_{n-2} = 2 \quad (10.31)$$

(i) The associated homogeneous equation is $a_n - 2a_{n-1} + a_{n-2} = 0$

Its characteristic equation is $r^2 - 2r + 1 = 0$

$\Rightarrow (r - 1)^2 = 0$

$\Rightarrow r = 1, 1.$

∴ The homogeneous solution is

$$a_n^{(H)} = (c_1 + c_2 n)1^n$$

(ii) Since the R.H.S. of the recurrence relation is 2, a constant, we assume the particular solution of the recurrence to be

$$a_n^{(H)} = d, \text{ a constant.}$$

Using this solution in the given recurrence relation, we get

$d - 2d + d = 2$

i.e., $0 = 2$, which is impossible.

Thus, our assumption is wrong.

Now, we assume that $a_n^{(p)} = nd$

Using this solution in the given recurrence relation, we get

$nd - 2(n - 1)d + (n - 2)d = 2$

$\Rightarrow nd - 2nd + nd + 2d - 2d = 2$

$\Rightarrow 0 = 2$, which is also impossible.

Thus, our assumption is wrong.

Now, we assume that $a_n^{(H)} = n^2 d$

Using this solution in the given recurrence relation, we get

$n^2 d - 2(n - 1)^2 d + (n - 2)^2 d = 2$

$\Rightarrow n^2 d - 2(n^2 - 2n + 1)d + (n^2 - 4n + 4)d = 2$

$\Rightarrow n^2 - 2n^2 d + 4nd - 2d + n^2 d - 4nd + 4d = 2$

$\Rightarrow d = 1$

$\therefore a_n^{(p)} = n^2$

Hence, the general solution is

$$a_n = a_n^{(H)} + a_n^{(p)}$$

$$\Rightarrow a_n = [c_1 + c_2 n]1^n + n^2 \quad (10.32)$$

Given $a_0 = 25$ and $a_1 = 16$.

Now, $a_0 = 25$

$\Rightarrow [c_1 + 0] + 0 = 25$

$\Rightarrow c_1 = 25$

Also, $a_1 = 16$

$\Rightarrow (c_1 + c_2) + 1^2 = 16$

$\Rightarrow c_1 + c_2 + 1 = 16$

$\Rightarrow c_2 = 16 - 1 - 25$

$= -10$

$\therefore$ The required solution is

$a_n = (25 - 10n)\, 1^n + n^2$

$\Rightarrow a_n = n^2 - 10n + 25$ ■

Example 10.21 Solve the recurrence relation

$$a_k - 7a_{k-1} + 10a_{k-2} = 6 + 8k,\ a_0 = 1,\ a_1 = 2$$

or solve

$$S(k) - 7S(k-1) + 10S(k-2) = 6 + 8k,\ S(0) = 1,\ S(1) = 2.$$

Solution The given recurrence relation is

$$a_k - 7a_{k-1} + 10a_{k-2} = 6 + 8k \quad (10.33)$$

(i) The associated homogeneous equation is

$a_k - 7a_{k-1} + 10a_{k-2} = 0$

Its characteristic equation is

$$r^2 - 7r + 10 = 0$$

$\Rightarrow (r-5)(r-2) = 0$

$\Rightarrow r = 2, 5$

Therefore, the homogeneous solution is

$$a_k^{(H)} = c_1 2^k + c_2 5^k$$

(ii) We need to find the particular solution,

let $a_k^{(P)} = d_0 + d_1 k$, since the R.H.S. is a linear polynomial.

Using this solution in the given recurrence relation, we get

$$(d_0 + d_1 k) - 7\,[d_0 + d_1\,(k-1)] + 10\,[d_0 + d_1\,(k-2)] = 6 + 8k$$

$\Rightarrow (d_0 - 7d_0 + 10d_0) + d_1\,[k - 7\,(k-1) + 10\,(k-2)] = 6 + 8k$

$\Rightarrow 4d_0 + d_1\,[k - 7k + 7 + 10k - 20)] = 6 + 8k$

$\Rightarrow (4d_0 - 13d_1) + 4d_1 k = 6 + 8k$

Equating the corresponding coefficients on both sides, we get

$$4d_0 - 13d_1 = 6 \text{ and } 4d_1 = 8$$

Now, $4d_1 = 8 \Rightarrow d_1 = 2$

$\Rightarrow 4d_0 = 6 + 13\,(2)$

$\Rightarrow d_0 = 8$

$\Rightarrow a_k^{(P)} = 8 + 2k$

Thus, the general solution is $a_k = a_k^{(H)} = a_k^{(P)}$

$\Rightarrow a_k = c_1 2^k + c_2 5^k + 8 + 2k$ (10.34)

Given that, $a_0 = 1$, $a_1 = 2$

Now, $a_0\ 1 \Rightarrow \quad c_1 + c_2 + 8 = 1$

$\Rightarrow c_1 + c_2 + 8 = -7$ (10.35)

Also, $a_1 = 2 \Rightarrow c_1 2 + c_2 5 + 8 + 2 = 2a$

$\Rightarrow 2c_1 + 5c_2 = -8$ (10.36)

Solving Eqs (10.35) and (10.36), we get

$$c_1 = -9 \text{ and } c_2 = 2$$

$\therefore$ The required solution is

$$a_k = -9\,(2^k) + 2\,(5^k) + 8 + 2k$$

Example 10.22 Solve the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2} + (n+1)\,2^n$$

Solution The given recurrence relation is

$$a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n \quad (10.37)$$

(i) The associated homogeneous equation is

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

Its characteristic equation is

$r^2 - 4r + 4 = 0$

i.e., $(r-2)^2 = 0$

$\Rightarrow r = 2, 2$

$\therefore$ The homogeneous solution is

$$a_n^{(H)} = (c_1 + c_2 n)2^n$$

(ii) Since the R.H.S. of the recurrence relation is $(n+1)2^n$ and 2, 2 is the characteristic root of the equation (i.e., 2 is repeated twice), we assume the particular solution of the recurrence relation to be

$$a_n^{(P)} = (c_1 + c_2 n)n^2 2^n$$

Using this solution in the given recurrence relation, we have

$(c_1 + c_2 n)n^2 2^n - 4\,[c + c_2\,(n-1)]\,(n-1)^2\,2^{n-1} + 4\,[c_1 + c_2\,(n-2)]\,(n-2)^2\,2^{n-2} = (n+1)2^n$

$\Rightarrow 4(c_1 + c_2 n)n^2 - 8(n-1)^2[c_1 + c_2(n-1)] + 4(n-2)^2[c_1 + c_2(n-2)] = 4(n+1)$

$\Rightarrow 4(c_1 + c_2 n)n^2 - 8(n^2 - 2n + 1)[c_1 + c_2(n-1)] + 4(n^2 - 4n + 4)[c_1 + c_2(n-2)]$
$= 4(n+1)$ (10.38)

Putting, $n = 0$, we get

$-8(c_1 - c_2) + 16(c_1 2c_2) = 4$

$\Rightarrow 8c_1 - 24c_2 = 4$

$\Rightarrow c_1 - 3c_2 = \frac{1}{2}$ (10.39)

Equating the coefficients of n on both sides of Eq. (10.38), we get

$$16c_1 - 16c_2 - 8c_2 - 16c_1 + 32c_2 + 16c_2 = 4$$

$\Rightarrow 24c_2 = 4$

$\Rightarrow c_2 = \frac{1}{6}$

Therefore, Eq. (10.39) gives

$$c_1 = \frac{1}{2} + \frac{1}{2}$$

$\Rightarrow c_1 = 1$

$$\therefore\ a_n^{(P)} = \left(1 + \frac{1}{6}n\right)n^2 2^n$$

$$= \left(n^2 + \frac{n^3}{6}\right)2^n$$

Thus, the general solution of the recurrence relation is

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$\Rightarrow a_n = \left[c_1 + c_2 n + n^2 + \frac{n^3}{6}\right]2^n$$

Example 10.23 Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 3n + 2^n$, $a_0 = 1$, $a_1 = 1$.

Solution The given recurrence relation is

$a_n - 4a_{n-1} - 4a_{n-2} = 3n + 2^n$ (10.40)

(i) The associated homogeneous equation is

$a_n - 4a_{n-1} - 4a_{n-2} = 0$

Its characteristic equation is $r^2 - 4r + 4 = 0$

$\Rightarrow (r-2)^2 = 0$

$\Rightarrow r = 2, 2$

Thus, the homogeneous solution is

$$a_n^{(H)} = (c_1 + c_2 n)2^n \tag{10.41}$$

(ii) R.H.S. = $3n + 2^n$

Particular solution = $a_n^{(P_1)} + a_n^{(P_2)}$

Since part of the R.H.S. is $3n$, i.e., a linear polynomial,

let $a_n^{(P_1)} = d_0 + d_1 n$

Using this solution in the recurrence relation, we get

$(d_0 + d_1 n) - 4\{d_0 + d_1(n-1)\} + 4\{d_0 + d_1(n-2)\} = 3n$

$\Rightarrow (d_0 - 4d_0 + 4d_0) + d_1[n - 4(n-1) + 4(n-2)] = 3n$

$\Rightarrow (d_0 - 4d_1) + d_1 n = 3n$

Equating the coefficients of n on both sides, we get

$$d_1 = 3$$

Equating the constant terms on both sides, we get

$$d_0 - 4d_1 = 0$$

$\Rightarrow d_0 = 12$

Therefore, particular solution corresponding to $3n$ is

$$a_n^{(P_1)} = 12 + 3n \tag{10.42}$$

Since part of the R.H.S. is 2^n and 2 is the double root of the characteristic equation, let us assume the particular solution to be $a_n^{P_2} = dn^2 2^n$

Using this solution in the given recurrence relation, we get

$dn^2 2^n - 4d(n-1)^2 2^{n-1} + 4d(n-2)^2 2^{n-2} = 2^n$

$\Rightarrow 4dn^2 - 8d(n-1)^2 + 4d(n-2)^2 = 4$

$\Rightarrow dn^2 - 2d(n-1)^2 + d(n-2)^2 = 1$

$\Rightarrow dn^2 - 2d(n^2 - 2n + 1) + d(n^2 - 4n + 4) = 1$

Putting $n = 0$, we get

$-2d + 4d = 1$

$\Rightarrow d = 1/2$

Therefore, the particular solution corresponding to 2^n is

$$a_n^{(P_2)} = \frac{1}{2} n^2 2^n$$

$$= n^2 2^{n-1} \tag{10.43}$$

Therefore, the particular solution is

$$a_n^P = a_n^{(P_1)} + a_n^{(P_2)}$$

$$= 12 + 3n + n^2 2^{n-1}$$

Hence, the general solution is

$a_n (c_1 + c_2 n)2^n + 12 + 3n + n^2 2^{n-1}$

Given that $a_0 = 1, a_1 = 1$

$a_0 = 1 \Rightarrow c_1 + 12 = 1$

$\Rightarrow c_1 = -11$

Also, $a_1 = -1 \Rightarrow (c_1 + c_2)\, 2 + 12 + 3 + 2^2 = 1$

$\Rightarrow 2c_1 + 2c_2 = -18$

$\Rightarrow c_1 + c_2 = -9$

$\Rightarrow c_2 = -9 - (-11) = 2$

$\Rightarrow c_2 = 2$

Thus, the required solution is

$$a_n = (2n - 11)\,2^n + 12 + 3n + n^2 2^{n+1}$$

Example 10.24 For what values of constants A and B is $a_k = Ak + B$ a solution of the recurrence relation $a_k = 2a_{k-1} + k + 10$.

Solution The given recurrence relation is $a_k = 2a_{k-1} + k + 10$. (10.44)

Now, $a_k = Ak + B$ is a solution of Eq. (10.44) if it satisfies the recurrence relation.

$\Rightarrow 2a_{k-1} + k + 5 = 2[A(k-1) + B] + k + 5$

$= (2A + 1)\,k + 2(B - A) + 5$

$= A\,k + B$, when $A = (2A + 1)$ and $2(B - A) + 5 = B$

When $A = -1$ and $B = -7$, the above relation holds true

Thus, for $A = -1$ and $B = -7$, $a_k = Ak + B$ is one of the solutions.

EXERCISES

10.9 Determine which of these equations are linear homogeneous recurrence relations with constant coefficients and find their degree.

(a) $a_k = a_{k-1}^2$ (b) $a_k = \dfrac{a_k - 1}{k}$

(c) $a_k = a_{k-1} + a_{k-2} + k + 3$

(d) $a_k = 4\,a_{k-2} + 5a_{k-4} + 9a_{k-7}$

10.10 Solve the recurrence relation $a_n = \dfrac{a_{n-2}}{4}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$.

10.11 Solve the recurrence relation $a_n = 7a_{n-2} + 6a_{n-3}$, $a_0 = 9$, $a_1 = 10$, $a_2 = 32$.

10.12 Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$, with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$ and $a_3 = 8$.

10.13 Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1$, $a_1 = 6$.

10.14 Solve the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, $a_0 = 2$, $a_1 = 5$ and $a_2 = 110$.

10.15 Solve $a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$, $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

10.16 Solve $a_n - a_{n-1} - 6a_{n-2} = -30$, with $a_0 = 20$ and $a_1 = -5$

10.17 Solve the recurrence relation $a_n - 3a_{n-1} - 4a_{n-2} = 4^n$.

10.18 Solve the recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n)$, $n \geq 0$. $a_0 = 1$, $a_1 = 4$.

10.19 Solve the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$, with $a_0 = 1$, $a_1 = 4$.

10.20 What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots 1, 1, 1, 1, –2, –2, –2, 3, 3, –4?