Recurrence Relation (UNIT - III)

Definition of Recurrence Relation

A recurrence relation is an equation that recursively defines a sequence or a rule that gives next term in the sequence as a function of the previous terms, when one or more initial terms are given.

Example:

The fibonacci sequence is defined by using the recurrence relation

$$\boxed{F_n = f_{n-1} + f_{n-2}}$$

with initial condition $\boxed{f_0 = 0,\ f_1 = 1}$

In the same way, we can find the next succeeding terms in the sequence such as.

$F_2 = F_1 + f_0$
$= 1 + 0$
$= 1$

$F_3 = F_2 + F_1$
$= 1 + 1$
$= 2.$

$F_4 = f_3 + f_2$
$= 2 + 1$
$= 3$

$F_5 = f_4 + f_3$
$= 3 + 2$
$= 5$

$f_6 = f_5 + f_4$
$= 5 + 3 = 8$

$f_7 = f_6 + f_5$
$= 8 + 5 = 13$

$f_8 = f_7 + f_6 = 13 + 8 = 21$

$\vdots$

$F_n = f_{n-1} + f_{n-2},\ n \geq 2.$

we obtain the sequence of fibonacci numbers by using the above recurrence relation.

$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

Ex: find out the sequence generating by the recurrence relation below. (b)

sol:

(a) $T_n = 2 \cdot T_{n-1}$ with $T_1 = 4$.

sol: Given Recurrence relation is.

$$\boxed{T_n = 2T_{n-1}}$$

Initial condition is: $T_1 = 4$.

$n=2$ $\quad T_2 = 2T_{2-1}$

$= 2T_1 = 2 \times 4 = 8$

$n=3$ $\quad T_3 = 2T_{3-1}$

$= 2T_2 = 2 \times 8 = 16.$

$n=4$ $\quad T_4 = 2T_{4-1}$

$= 2T_3 = 2 \times 16 = 32$

$n=5$ $\quad T_5 = 2T_{5-1}$

$= 2T_4 = 2 \times 32 = 64.$

n:	2	3	4	5
T_n:	8	16	32	64.

Recursive sequence generated by the recurrence relation is: 4, 8, 16, 32, 64 ---

(b) $T_n = 3 \cdot T_{n-1} - 4$ with $T_1 = 3$.

Sol:- Given recurrence relation is.

$$T_n = 3T_{n-1} - 4.$$

Initial condition is $T_1 = 3$.

$n=2 \Rightarrow T_2 = 3 \cdot T_{2-1} - 4$

$= 3 \cdot T_1 - 4$

$= 3*3-4 = 9-4 = 5.$

$n=3 \Rightarrow T_3 = 3 \cdot T_{3-1} - 4$

$= 3 \cdot T_2 - 4$

$= 3*5 - 4 = 15-4 = 11$

$n=4 \Rightarrow T_4 = 3 \cdot T_{4-1} - 4$

$= 3 T_3 - 4$

$= 3*11 - 4 = 33-4 = 29.$

$n=5 \Rightarrow T_5 = 3 \cdot T_{5-1} - 4$

$= 3 \cdot T_4 - 4.$

$= 3*29 - 4 = 87-4 = 83.$

Recursive sequence $= 3, 5, 11, 29, 83 \ldots$

(3) Find out the recurrence relation for the following sequence.

(a) 2, 6, 18, 14, 162 ....

(b) 20, 17, 14, 11, 18 ....

(c) 1, 3, 6, 10, 15, 21 ---

(a) 2, 6, 18, 54, 162 -----

Sol:- Given recursive sequence = 2, 6, 18, 54, 162 ...

T_1, T_2, T_3, T_4, T_5 is the recursive sequence.

$T_1 = 2$ $\quad$ $T_4 = 54$

$T_2 = 6$ $\quad$ $T_5 = 162$

$T_3 = 18$

This is the initial conditions in the sequence.

Initial condition is: $T_1 = 2$.

$T_2 = 6 = 3 \cdot T_1 = 3 \cdot 2 = 6.$ $\quad$ $T_2 = 6 = 3 \cdot T_1 = 3 \cdot 2 = 6$

$T_3 = 18 = 3 \cdot T_2 = 3 \cdot 6 = 18.$ $\quad$ $T_3 = 18 = 3 \cdot T_2 = 3.$

$T_4 = 54 = 3 \cdot T_3 = 3 \cdot 18 = 54$

$T_5 = 162 = 3 \cdot T_4 = 3 \cdot 54 = 162.$ $\quad$ $\boxed{T_n = 3 \cdot T_{n-1}}$

$\vdots$

$T_n = 3 \cdot T_{n-1}.$ with $T_1 = 2.$

$\therefore$ Recurrence Relation for the sequence, ~~...~~

2, 6, 18, 54, 162 ---- is $\boxed{T_n = 3T_{n-1}}$ with $T_1 = 2.$

Prove that.

(b) 20, 17, 14, 11, 8 ----

Sol:- Given recurrence ~~relation~~ sequence is

= 20, 17, 14, 11, 8 ----.

T_1, T_2, T_3, T_4, T_5 is the recursive sequence.

$T_1 = 20$ $\quad$ $T_3 = 14$ $\quad$ $T_5 = 8.$

$T_2 = 17$ $\quad$ $T_4 = 11$

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Initial Conditions $T_1 = 20$.

$\cancel{T_2} = \cancel{3}$

$T_2 = 17 = T_1 - 3 = 20 - 3 = 17$.

$T_3 = 14 = T_2 - 3 = 17 - 3 = 14$.

$T_4 = 11 = T_3 - 3 = 14 - 3 = 11$.

$T_5 = 8 = T_4 - 3 = 11 - 3 = 8$.

⋮

$\boxed{T_n = T_{n-1} - 3}$ with initial condition $T_1 = 20$.

(c) 1, 3, 6, 10, 15, 21 . . .

Sol: Given ~~relation~~ sequence is 1, 3, 6, 10, 15, 21.

Given recurrence relation sequence.

T_1, T_2, T_3, T_4, T_5 is the recursive sequence.

$T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, $T_5 = 15$, $T_6 = 21$.

Initial conditions $T_1 = 1$.

$T_2 = 3 = T_1 + 2 = 1 + 2 = 3$.

$T_3 = 6 = T_2 + 3 = 3 + 3 = 6$.

$T_4 = 10 = T_3 + 4 = 6 + 4 = 10$.

$T_5 = 15 = T_4 + 5 = 10 + 5 = 15$

$T_6 = 21 = T_5 + 6 = 15 + 6 = 21$.

⋮

$\boxed{T_n = T_{n-1} + n}$ With initial condition $T_1 = 1$.

Note:- The next term depends on previous one term called first order relation.

Example: Find the first five terms of the sequence defined by each of the following recurrence relation and initial conditions.

(a) $a_n = a_{n-1}^2$, $a_1 = 2$.

The given recurrence relation is $a_n = a_{n-1}^2$.

Given that $a_1 = 2$.

$$a_2 = a_{2-1}^2$$
$$= a_1^2 = 2^2 = 4$$

$$a_3 = a_{3-1}^2$$
$$= a_2^2 = 4^2 = 16.$$

$$a_4 = a_{4-1}^2$$
$$= a_3^2 = 16^2 = 256.$$

$$a_5 = a_{5-1}^2$$
$$= a_4^2 = (256)^2 = 65,356.$$

Hence the first five term of sequence are

2, 4, 16, 256 and 65,356.

(b) $a_n = n a_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$.

Sol:- Given recurrence relation

~~$a_n =$~~

$$\boxed{a_n = n a_{n-1} + n^2 a_{n-2}}$$

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Initial Conditions. $a_0 = 1,\ a_1 = 1.$ 3.4

$a_2 = 2a_{2-1} + 2^2 a_{2-2}$

$= 2a_1 + 4a_0.$

$= 2\times1 + 4\times1 = 2+4 = 6.$

$a_3 = 3a_{3-1} + 3^2 a_{3-2}.$

$= 3a_2 + 9a_1$

$= 3\times6 + 9\times1 = 18+9 = 27.$

$a_4 = 4a_{4-1} + 4^2 a_{4-2}$

$= 4a_3 + 16a_2$

$= 4\times27 + 16\times6 = 204.$

Thus the first five terms of the sequence are.

1, 1, 6, 27 and 204.

(2) $a_n = a_{n-1} + a_{n-3}$ $\quad a_0 = 1,\ a_1 = 2,\ a_2 = 0.$

Sol: Given recurrence relation.

$$\boxed{a_n = a_{n-1} + a_{n-3}}$$

Initial Conditions $a_0 = 1,\ a_1 = 2,\ a_2 = 0.$

$a_3 = a_{3-1} + a_{3-3}$

$= a_2 + a_0 = 0+1 = 1.$

$a_4 = a_{4-1} + a_{4-3} = a_3 + a_1 = 1+2 = 3.$

$a_5 = a_{5-1} + a_{5-3} = a_4 + a_2 = 3+0 = 3.$

Thus, the first five terms of the sequence are

1, 2, 0, 1, 2, 4 ...

Ex: let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3a_{n-2}$ for $n = 2, 3, \ldots$ and let $a_0 = 1$ and $a_1 = 2$. What are the values of a_2 and a_3.

Sol:- The given recurrence relation is

$$\boxed{a_n = a_{n-1} + 3a_{n-2}}, \quad n = 2, 3 \ldots$$

$$a_2 = a_{2-1} + 3a_{2-2}.$$

$$= a_1 + 3a_0 = 2 + 3 \times 1 = \cancel{2\times} 2 + 3 = 5.$$

$$\therefore a_2 = 5.$$

$$a_3 = a_{3-1} + 3a_{3-2}$$

$$= a_2 + 3a_1 = 5 + 3 \cdot 2 = 5 + 6 = 11.$$

$$\therefore a_3 = 11.$$

Example: Determine whether the sequence $\{a_n\}$ is a solution of the recurrence relation
$a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if.

(i) $a_n = -n + 2$ (ii) $a_n = 3(-1)^n + 2^n - n + 2$.

Sol: The given recurrence relation is

$$\boxed{a_n = a_{n-1} + 2a_{n-2} + 2n - 9.}$$

(i) $a_n = -n + 2$.

$$a_{n-1} = -(n-1) + 2$$

$$a_{n-2} = -(n-2) + 2.$$

$= a_{n-1} + 2a_{n-2} + 2n - 9.$

$= [-(n-1)+2] + 2[-(n-2)+2] + 2n - 9.$

$= -n+1+2+2\{-n+2+2] + 2n - 9$

$= -n+1+2 - \cancel{2n} + 4 + 4 + \cancel{2n} - 9.$

$= -n+3+8-9$

$= -n+11-9$

$= -n+2.$

$= a_n. = L.H.S.$

$\therefore a_n = -n+2$ is a solution of the recurrence relation.

(ii) $a_n = 3(-1)^n + 2^n - n + 2.$

$R.H.S = a_{n-1} + 2a_{n-2} + 2n - 9.$

$a_{n-1} = 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2.$

$a_{n-2} = 3(-1)^{n-2} + 2^{n-2} - (n-2) + 2.$

$= 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2 + 2(3(-1)^{n-2} + 2^{n-2} - (n-2) + 2] + 2n - 9.$

$= 6(-1)^{n-2} + 3(-1)^{n-1} + 2(2^{n-2}) + 2^{n-1} - n + 1 + 2 - 2n + 4 + 4 + 2n - 9.$

$= 6(-1)^{n-2} + 3(-1)^{n-1} + 2\cdot(2^{n-2}) + 2^{n-1} - n + 2.$

$= 6 \cdot \frac{(-1)^n}{(-1)^2} + 3 \cdot \frac{(-1)^n}{(-1)^1} + 2 \cdot \frac{2^n}{2^2} + \frac{2^n}{2^1} - n + 2$

$= 6(-1)^n - 3(-1)^n + \frac{2^n}{2} + \frac{2^n}{2} - n + 2$

$\cancel{= 6(-1)^n - 3(-1)^n + 2 \cdot \frac{2^n}{2} = n + 2}$

$\cancel{= 6(-1)^n - 3(-1)^n - n + 2}$

$= 6(-1)^n - 3(-1)^n + \cancel{2} \cdot \frac{2^n}{\cancel{2}} - n + 2$

$= 6(-1)^n - 3(-1)^n + 2^n - n + 2.$

$= 3(-1)^n + 2^n - n + 2.$

$= a_n. =$ L.H.S.

$\therefore a_n = 3(-1)^n + 2^n - n + 2$ is a solution of the given recurrence relation.

<u>Example:</u>

$\Rightarrow$ let $a_n = 2^n + 5(3^n)$, for $n = 0, 1, 2 \ldots$

(i) Find a_0, a_1, a_2, a_3 and a_4.

(ii) show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$ and $a_4 = 5a_3 - 6a_2$.

(iii) ST $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \geq 2$.

Solve the recurrence relations

$$a_n = 5a_{n-1} - 6a_{n-2} \text{ for } n \geq 2, \; a_0 = 1, \; a_1 = 0.$$

Sol: The given recurrence relation is

$$a_n - 5a_{n-1} + 6a_{n-2} = 0.$$

The characteristics equation of the recurrence relation

is $r^2 - 5r + 6 = 0.$

Characteristic roots: $r(r-2) + 3(r+1).$

$$r(r-3) - 2(r-3)$$

$$r^2 - 3r - 2r + 6.$$

$$\Rightarrow r(r-3) - 2(r-3)$$

$$r - 2 = 0 \Rightarrow r = 2$$

$$r - 3 = 0 \Rightarrow r = 3.$$

Hence, the solution is

$$a_n = c_1 \cdot 2^n + c_2 \cdot 3^n.$$

Initial conditions are $a_0 = 1, \; a_1 = 0.$

now $a_0 = c_1 \cdot 2^0 + c_2 \cdot 3^0.$

$$\Rightarrow c_1 + c_2 = 1. \quad \text{—— (1)}$$

$$a_1 = c_1 \cdot 2^1 + c_2 \cdot 3^1$$

$$2c_1 + 3c_2 = 0. \quad \text{—— (2)}$$

Solving (1) & (2): $c_1 = 1 - c_2.$

$$2(1 - c_2) + 3c_2 = 0.$$

$$2 - 2c_2 + 3c_2 = 0$$

$$2 + c_2 = 0 \Rightarrow c_2 = -2.$$

$$c_1 = 1 + 2 = 3$$

Hence the sol. is

$$a_n = 3 \cdot (2^n) - 2 \cdot (3^n).$$

for $n \geq 2.$

Example Solve the recurrence relation is

$$a_n - 8a_{n-1} + 16a_{n-2} = 0. \text{ for } n \geq 2, a_0 = 16, a_1 = 80$$

Sol:- The given recurrence relation is.

$$a_n - 8a_{n-1} + 16a_{n-2} = 0$$

The characteristic equation is

$$r^2 - 8r + 16 = 0.$$

Characteristic roots.

$$r(r-4) \cancel{+} - 4(r-4).$$

$$\Rightarrow (r-4)^2 = 0.$$

$$\Rightarrow r = 4, 4.$$

roots are equal.

$$a_n = (c_1 + c_2 n) r^n.$$

Hence the solution is.

$$\boxed{a_n = \cancel{4} \, c_1 4^n + c_2 \overset{n}{\times} 4^n.}$$ where c_1 & c_2 are arbitrary constants.

Initial conditions:

$$a_0 = 16 \Rightarrow c_1 \cdot 4^0 + c_2 \cdot 4^0 \times 0$$

$$\Rightarrow c_1 + 0 = 16. \Rightarrow c_1 = 16.$$

$$a_1 = 80 \Rightarrow c_1 \cdot 4^1 + c_2 \cdot 4^1 \times 1.$$

$$\Rightarrow 4c_1 + 4c_2 = 80.$$

$$\Rightarrow 4 \times 16 + 4c_2 = 80$$

$$4c_2 = 80 - 64$$

$$4c_2 = 16 \Rightarrow c_2 = 16/4 = 4.$$

16
4
64
16

Hence the unique solution is

$$a_n = 16(4^n) + 4n(4^n) = 4^{n+2} - n4^{n+1}$$

$$a_n = 4^1 * 4^{n+1} - n4^{n+1}$$

$$= 4^{n+1}(4-n), \; n \geq 2.$$

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