# Discrete Mathematics (UNIT-I)

## What is discrete mathematics.

**Definition:**

Discrete mathematics is mathematics that deal with discrete objects. Discrete objects are those which are separated from (not connected to/ distinct from) each other.

i.e objects that can assume only distinct, separated values.

- The term "Discrete mathematics" is therefore used in contrast with "continuous mathematics" which is the branch of mathematics dealing with objects.

**Examples of objects with discrete values**

- Integers, graphs/ statements in logic, automobiles, peoples, houses etc.

**Discrete mathematics and Computer Science**

- concepts from discrete mathematics are useful for describing objects and problems in computer algorithms and problems programming languages.
- These have applications in cryptography, automated theorem proving, and software development.

**Why discrete mathematics:**

Let us first see why we want to be interested the formal/ theoretical approaches in computer sci

Some of the major reasons that we adopt t[illegible]
approaches are.

1.) we can handle infinity or large quantity and indefiniteness with them.

2) results from formal approaches are reusable.

## Standard applications of Discrete mathematics in computer science:

### UNIT-I: (propositional logic).

① The design of digital circuits is entirely based on propositional logic, so much so that its engineers call it "logic design" rather than "circuit design"

② Even writing a computer program is often thought to ~~involu-~~ involve devising its "logic".

(note that "logic" in the later sense is an informal logic), used to refer to the idea rather than formal flow on information through the program and whether it is being processed correctly.

# Mathematical Logic:

What is logic –

Mathematical or propositional logic. –

↓

The area of logic that deals with the ~~prop~~ proposition.

## Contingency

A stat formulae which is neither a tautology nor a contradiction is called contingency.

$(p \to q) \wedge (p \wedge q)$ is a contingency.

Ex: tautology or contingency or contradiction

i) $p \to (q \to p)$ — T

ii) $(q \wedge p) \vee (q \vee \sim p)$ — C

iii) $q \vee (\sim q \wedge p)$ — C.

PT

## PTL

<u>logic</u>: It deals with methods of reasoning.

The main aim of logic is to provide rules by which we can determine whether the particular reasoning / argument is valid.

✓ Greek philosopher, Aristotle was the pioneer of the logical reasoning.

<u>Def. proposition</u>: It is a collection of declarative stmt that has either a truth values, "true" or "false" can be assigned but not both.

✓ a, b, c, --- d —— alphabets.

✓ cat, dog, man, chair —— words.

✓ She is a girl ——
He is a boy —— } sentence.

Sentences
- Declarative — either T/F.
- Imperative — no T/F.
- Interrogative —(?)
- Exclamatory. —(!).

~~Mostly~~ Mostly lower case letters can be used to represent the propositions.

# Mathematical Logic (UNIT - I)

## Mathematical or propositional logic

The rules of mathematical logic specify methods of reasoning mathematical statements.

(or)

Logic is the discipline that deals with methods of reasoning. The main aim of logic is to provide rules by which we can determine whether the particular reasoning or argument is valid.

→ Greek philosopher, Aristotle was the pioneer of logical reasoning.

→ Logical reasoning is used to in many disciplines to establish valid results.

**Rules of logic are used to provide :-**

1. proofs of theorems in mathematics
2. to verify the correctness of computer programs.
3. to draw the conclusions from scientific experiments.

→ It has many practical applications in computer science like design of computing machines, artificial Intelligence, definition of data structures for programming languages etc

**Propositions or statements :-**

**Definition: (propositional logic)**

It is concerned with statements to which the truth values, "true" or "false" can be assigned.

The purpose is to analyze these statements either individually or in a composite manner.

Definition:- (Proposition)

A proposition is a Collection of declarative statements that has either a truth value "true" (or) "false" can be assigned but not both.

→ sentences which are exclamatory, interrogative or imperative in nature are not propositions.

→ Lower case letters such as p, q, r --- are used to denote the propositions.

For example; we consider the following sentences.

Valid propositions:

1. New Delhi is the capital city of india (T)
2. $2+2=3$ (F)
3. Man is mortal. (T)
4. "$1+2+9 = 3-2$" (F).

invalid propositions:

1. How beautiful is rose? (interrogative)
2. Take a cup of coffee. (imperative)
3. "A is less than 2" (It is because unless we give a specific value of A, we can not say whether the statement is true or false.).

→ If the proposition is true, we say that the truth value of that proposition is true. denoted by "T" or "1".

→ If the proposition is false, the truth value is said to be false. denoted by "F" or "0".

There are two types of Propositions:

✓ ① Primitive or Primary or atomic

✓ ② compound or molecular proposition

① Primitive or Primary:

Proposition which do not contain any of the logical operator or connectives are called atomic or primitive or primary proposition.

② Compound or molecular:

Combine two or more atomic statements using connectives are called molecular or compound statement.

→ propositional Logic:

The area of logic that deals with propositions is called propositional calculus or propositional logic.

Note:

The truth value of a compound statement or proposition depends on those of sub propositions. And the way in which they are combined using connectives.

Connectives:

There are five types of connectives.

① Conjunction

② disjunction

③ conditional

④ biconditional

⑤ negation

① Conjunction:

Let p and q are two statements. The statement $p \wedge q$ is called conjunction of the "p and q". which is read as p and q.

The statement has the truth value T when both p and q have the truth value T. Otherwise it is False (F).

Truth table:-

Definition: Determine the truth value of statement formula for each possible combination of the truth values of the compound statement.

A table showing all such truth values is called the truth table of the formula.

Truth table for Conjunction:

| p | q | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

Example:-

p: Canada is a country

q: moscow is the capital of spain.

$p \wedge q$: Canada is a country "and" moscow is the capital of spain.

Ex: p: Today is Monday

q: There are 50 tables in this room.

The conjunction of p and q, that is $p \wedge q$ may be written as

$p \wedge q$: Today is monday and there are 50 tables in this room.

② disjunction:

The disjunction of the statements p, q is denoted by $p \vee q$. which is read as "p or q"

The statement $p \vee q$ has ~~two~~ the truth value F only when both p and q have the truth value F. Otherwise it is T.

Truth table for disjunction:

| p | q | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

Example:

① $p \vee q$: Canada is a country or moscow is the capital of spain.

② $p \vee q$: Today is monday or there are 50 tables in this room.

## ③ Conditional or implication;

The implication of two statements $p, q$ is denoted by $p \to q$.

which is read as ① if $p$ then $q$

② $p$ is sufficient for $q$.

③ $p$ is sufficient condition for $q$

④ $q$ is necessary for $p$.

⑤ $q$ is necessary condition for $p$.

⑥ $p$ only if $q$. etc.

→ The statements $p \to q$ has a truth value 'F' when $p$ has truth value T and $q$ has truth value F. otherwise it is T.

**Truth table for implication:**

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

The verbal translation of $p \to q$ is.

① $p \to q$: If canada is a country then moscow is the capital of spain.

② $p \to q$: If today is monday then there are 50 tables in this room.

Note: The statement '$p$' is called "hypothesis" of the implication. And $q$ is called "conclusion".

## (4) Biconditional or bi-implication:



If p, q are two statements then the biconditional of two statements p, q are denoted by $p \leftrightarrow q$ or $p \rightleftarrows q$.

which is read as (1) p if and only if q

(2) p iff q

(3) p is necessary and sufficient for q.

(4) If p then q, and conversely.

→ The statement $p \leftrightarrow q$ has the truth value T when both p and q have identical truth values. Otherwise it has a truth value F.

Truth table for biconditional:

| p | q | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| F | F | F |
| F | T | F |
| F | F | T |

Example:

(1) $p \leftrightarrow q$: Canada is a country if and only if moscow is a capital of spain.

(2) $p \leftrightarrow q$: Today is monday if and only if there are 50 tables in this room.

(5) Negation:

If $p$ is a statement then its negation is $\neg p$ or $\sim p$.

which is read as "not p".

→ If the truth value of $p$ is T, then truth value of $\sim p$ is F. also if ~~tat~~ truth value of $p$ is F, then the truth value of $\sim p$ is T.

Truth table for negation:

| $p$ | $\sim p$ |
|---|---|
| T | F |
| F | T |

Example:

(1) $p$: Canada is a country

$\sim p$: Canada is not a country.

(2) $p$: chennai is a city.

$\sim p$: It is not the case that chennai is a city.

## Order of precedence for logical operators:

Order of precedence for the logical operators given as follows

① the negation operator has precedence over all the logical operators. thus

$\sim p \wedge q$ means $(\sim p) \wedge q$ but not $\sim (p \wedge q)$.

② the conjunction operator has precedence over the disjunction operator. Thus

$p \wedge q \vee r$ means $(p \wedge q) \vee r$, but not $p \wedge (q \vee r)$

③ the conditional and biconditional operators $\rightarrow$ and $\leftrightarrow$ have lower precedence than other operators. among them, $\rightarrow$ has precedence over $\leftrightarrow$.

Note: $(p \vee q) \wedge (\sim r)$ is the conjunction of $p \vee q$ and $\sim r$.

construct the truth table for $(p \rightarrow q) \wedge ((q \wedge \sim r) \rightarrow (p \vee r))$

| $p$ | $q$ | $r$ | $p \rightarrow q$ ① | $\sim r$ | $q \wedge \sim r$ | $p \vee r$ | $q \wedge \sim r \rightarrow p \vee r$ ② | ① $\wedge$ ② |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |

Truth table for $\sim(p \vee (q \wedge r)) \longleftrightarrow (\ldots$

Construct the truth table for $\sim(p \wedge q) \longleftrightarrow \sim p \vee \sim q$.

| $p$ | $q$ | $p \wedge q$ | $\sim(p \wedge q)$ ① | $\sim p$ | $\sim q$ | $\sim p \vee \sim q$ ② | ① $\longleftrightarrow$ ② |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |

→ Construct the truth table for the following formulas.

(i) $(q \wedge (p \to q)) \to p$

(ii) $\sim(p \vee (q \vee r)) \longleftrightarrow (p \vee q) \wedge (p \vee r)$.

Truth table for $(q \wedge (p \to q)) \to p$

| $p$ | $q$ | $p \to q$ | $q \wedge (p \to q)$ | $(q \wedge (p \to q)) \to p$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 |

1.6

Truth table for $\sim(P \vee (q \wedge r)) \leftrightarrow (P \vee q) \wedge (P \vee r)$

| $P$ | $q$ | $r$ | $q \wedge r$ | $P \vee (q \wedge r)$ | $\sim(P \vee (q \wedge r))$ ① | $P \vee q$ | $P \vee r$ | $(P \vee q) \wedge (P \vee r)$ ② | ① $\leftrightarrow$ ② |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |

## Well-Formed Formulas:

4, 5, 6,

**Definition:- (statement formula)**

A statement formula is an expression which is a string consisting of variables, parentheses, and connective symbols. Note that every string of these symbols is a statement formula.

We shall now give a recursive definition of a statement formula, often called a well formed formula (wff).

A well formed formula can be generated by the following rules

$R_1$: A statement variable standing alone is a well formed formula.

$R_2$: If $A$ is well formed formula, then $\sim A$ is a well formed formula.

If A and B are well formed formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, and $(A \leftrightarrow B)$ are well formed formulas.

Definition: (well formed formula)

A string of symbols containing the statement variables, connectives, and parenthesis is said to be well formed formula if it can be obtained by finitely many applications of rules $R_1, R_2$ and $R_3$.

Ex:- According to this definition, the following are well formed formulas.

$\sim(P \wedge Q)$, $\sim(P \vee Q)$, $(P \to (P \vee Q))$, $(P \to (Q \to R))$ and

$(((P \to Q) \wedge (Q \to R)) \rightleftarrows (P \to R))$.

Example 1:-

$\sim P \wedge Q$ is not a well formed formula because a wff would be either $(\sim P \wedge Q)$ or $\sim(P \wedge Q)$.

Here $\sim P \wedge Q$ does not contain parenthesis.

Example 2:-

$(P \to Q$ is not well formed formula.

but $(P \to Q)$ is a well formed formula (wff).

Example 3:-

$(P \wedge Q) \to Q)$ is not wff. because one of the parenthesis in the beginning is missing.

Example 4:-

$(P \to (P \vee Q))$ is a well formed formula.

Example 5:-

$((P \to (\sim P)) \to (\sim P))$ is a wff.

Example 6:

$$((P \rightarrow (Q \rightarrow R))) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R)))$$ is not wff.

because one of the parenthesis in the begening is missing.

Example 7:-

$$((\sim P \rightarrow Q) \rightarrow (Q \rightarrow P)))$$

It is not wff because one more parenthesis in the end.

Example 8:-

$((P \wedge Q) \Rightarrow)$ is a wff.

- Tautologies and Contradictions

It is compornd statement that is always tre.

→ the truth table of a tautology will contain only 'T' entries in the last column.

Importance of tautology:

- Tautology helps conclude some statement from some given statements.

For Example:-

if the statements "$p \Rightarrow q$" is a tautology, then it is easy to conclude the truth of $q$ from the truth of $p$.

Thus with the help of tautology we move from some given statement to some concluding statement in a step-by-step manner.

which is justified with in the frame work of mathematical logic.

Example: Show That $p \vee \sim p$ is a tautology.

| $p$ | $\sim p$ | $p \vee \sim p$ |
|---|---|---|
| T | F | T |
| F | T | T |

$p \vee \sim p$ has truth value 'T' for all its entries in the false.

$\therefore$ it is a ~~tautology~~. tautology. This is also called "Law of enclusive middle". i.e either p is true or it is false. there is no middle possibility.

Example: Show that statement $(p \wedge q) \Rightarrow q$ is a tautology.

| $p$ | $q$ | $p \wedge q$ | $(p \wedge q) \rightarrow q$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

$\therefore$ All entries in the last column has truth value

$\therefore$ it is a tautology.

## contradiction:

A Statement is said to be contradiction its truth value is always false (F) for all its entries in the truth table.

→ If the statement is a contradiction, then its negation will be a tautology.

Example: Show that $p \wedge \sim p$ is a tautology or contraction.

| $p$ | $\sim p$ | $p \wedge \sim p$ |
|---|---|---|
| T | F | T |
| F | T | T |

∴ since the statement $p \wedge \sim p$ has its false value for all its entries in the truth table.

Hence it is a contradiction.

Example: Establish $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$ as a tautology.

| $p$ | $q$ | $p \wedge q$ | $\sim(p \wedge q)$ | $\sim p$ | $\sim q$ | $\sim p \vee \sim q$ | $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F | T |
| T | F | F | T | F | T | T | T |
| F | T | F | T | T | F | T | T |
| F | F | F | T | T | T | T | T |

∴ the given statement is a tautology.

Examples:- (Exercises for student)

① Show that $(p \wedge q) \Rightarrow (p \vee q)$ is tautology but $(p \vee q) \Rightarrow (p \wedge q)$ is not.

② Show that $p \vee \sim(p \wedge q)$ is a tautology.

③ Show that $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction or tautology.

④ Make the truth table for $(p \downarrow q) \wedge (p \downarrow r)$, $(p \uparrow q \uparrow r)$ and $(p \oplus q \oplus r)$.

Equivalence of propositions (or) Logically equivalent (or) Equivalence of formulas

Two compound statements $A(p_1, p_2, \ldots p_n)$ and $B(p_1, p_2, \ldots p_n)$ are said to be logically equivalent or simply equivalent if they have identical truth tables.

If the truth value of A is ~~equival~~ equival to the truth value of B for every one of $2^n$ possible sets of truth values assigned to $p_1, p_2, p_3 \ldots p_n$.

The equivalence of two propositions A and B is denoted as $A \Leftrightarrow B$ or $A \equiv B$.

which is read as "A is equivalent to B".

Note:- $\Leftrightarrow$, $\equiv$ these are not a connectives.

## (a) Truth table method:

Ex: Prove that $(p \leftrightarrow q) \Leftrightarrow (\sim p \vee q) \wedge (\sim q \vee p)$

| $p$ | $q$ | $p \leftrightarrow q$ | $\sim p$ | $\sim p \vee q$ (1) | $\sim q$ | $\sim q \vee p$ (2) | (1) $\wedge$ (2) |
|---|---|---|---|---|---|---|---|
| T | T | T | F | T | F | T | T |
| T | F | F | F | F | T | T | F |
| F | T | F | T | T | F | F | F |
| F | F | T | T | T | T | T | T |

$\therefore p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$.

Example: Prove that $p \wedge (\sim q \vee r)$ and $p \vee (q \wedge \sim r)$ are logically equivalent or not.

Sol: $p \wedge (\sim q \vee r) \equiv p \vee (q \wedge \sim r)$.

| $p$ | $q$ | $r$ | $\sim q$ | $\sim q \vee r$ | (1) $p \wedge (\sim q \vee r)$ | $\sim r$ | $q \wedge \sim r$ | (2) $p \vee (q \wedge \sim r)$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | T | T | F | F | T |
| T | T | F | F | F | F | T | T | T |
| T | F | T | T | T | T | F | F | T |
| T | F | F | T | F | T | T | F | T |
| F | T | T | F | T | F | F | F | F |
| F | T | F | F | F | F | T | T | T |
| F | F | T | T | T | F | F | F | F |
| F | F | F | T | T | F | T | F | F |

$\therefore p \wedge (\sim q \vee r) \not\equiv p \vee (q \wedge \sim r)$.

Example: Prove that the following equivalences are.

ⓐ $\sim(p \to q) \equiv p \wedge \sim q$.

ⓑ $(p \vee q) \wedge (p \vee \sim q) \equiv p$.

ⓒ $(p \wedge q) \vee (p \wedge \sim q) \equiv p$.

ⓐ

| $p$ | $q$ | $p \to q$ | $\sim(p \to q)$ | $\sim q$ | $p \wedge \sim q$ |
|---|---|---|---|---|---|
| T | T | T | F | F | F |
| T | F | F | T | T | T |
| F | T | T | F | F | F |
| F | F | T | F | T | F |

Equivalent

ⓑ

| $p$ | $q$ | $p \vee q$ | $\sim q$ | $p \vee \sim q$ | $(p \vee q) \wedge (p \vee \sim q)$ |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | F | T | T | T | T |
| F | T | T | F | F | F |
| F | F | F | T | T | F |

Equivalent

ⓒ

| $p$ | $q$ | $p \wedge q$ | $\sim q$ | $p \wedge \sim q$ | $(p \wedge q) \vee (p \wedge \sim q)$ |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| T | F | F | T | T | T |
| F | T | F | F | F | F |
| F | F | F | T | F | F |

Equivalent

## (b) Replacement process:

Consider a formula $A: P \to (Q \to R)$. The formula $Q \to R$ is a part of the formula $A$.

If we replace $Q \to R$ by an equivalent formula $\sim Q \vee R$ in $A$. we get another formular $B: P \to (\sim Q \vee R)$.

one can easily verify that the formulas $A$ & $B$ are equivalent to each other.

This process of obtaining $B$ from $A$ is known as the replacement process.

Ex:- Prove that $P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$. ✓

Sol:- Replacing $Q \to R$ by $\sim Q \vee R$. we get

$$P \to (Q \to R) = P \to (\sim Q \vee R)$$ which is equivalent to

$$= \sim P \vee (\sim Q \vee R)$$

$$= \sim P \vee \sim Q \vee R$$

$$= (\sim P \vee \sim Q) \vee R$$

$$= \sim (P \wedge Q) \vee R.$$

$$= (P \wedge Q) \to R.$$

Ex: Prove that $(P \to Q) \wedge (R \to Q) \Leftrightarrow (P \vee R) \to Q$.

Sol: $(P \to Q) \wedge (R \to Q) \Leftrightarrow (\sim P \vee Q) \wedge (\sim R \vee Q)$

$\Leftrightarrow (\sim P \wedge \sim R) \vee Q$

$[\because (S_1 \vee S_2) \wedge (S_3 \vee S_2) \Leftrightarrow (S_1 \wedge S_3) \vee S_2]$

$\Leftrightarrow \sim (P \vee R) \vee Q$.

$\Leftrightarrow (P \vee R) \to Q$.

Ex: $P \to (Q \to P) \Leftrightarrow \sim P \to (P \to Q)$. ✓

Sol: $P \to (Q \to P) \Leftrightarrow \sim P \vee (Q \to P)$.

$\Leftrightarrow \sim P \vee (\sim Q \vee P)$.

$\Leftrightarrow \sim P \vee \sim Q \vee P$.

$\Leftrightarrow (\sim P \vee P) \vee \sim Q$.

$\Leftrightarrow T \vee \sim Q$.

$\Leftrightarrow T$. (L.H.S).

$\sim P \to (P \to Q) \Leftrightarrow \sim (\sim P \vee (P \to Q))$.

$\Leftrightarrow \sim (\sim P \vee (\sim P \vee Q))$.

$\Leftrightarrow P \vee (\sim P \vee Q)$

$\Leftrightarrow (P \vee \sim P) \vee Q$.

$\Leftrightarrow T \vee Q$

$\Leftrightarrow T$. (R.H.S.).

L.H.S = R.H.S.

so $P \to (Q \to P) \Leftrightarrow \sim P \to (P \to Q)$.

## Converse, contrapositive and inverse

we can arrange some new conditional statements using a conditional statements. $p \to q$.

**Converse:-**

Converse of the $p \to q$ is proposition $q \to p$.

**Contrapositive:-**

The contrapositive of $p \to q$ is the proposition $\sim q \to \sim p$.

**inverse:-**

The inverse of $p \to q$ is the proposition $\sim p \to \sim q$.

**Note:-** The above three conditional statements formed from $p \to q$.

**Example:-**

where $p, q$ represents the statements.

p: Priya is concerned about her cholesterol levels.

q: priya walks at least two miles three times a week.

① **Implication** ($p \to q$).

If priya is concerned about her cholesterol levels then she will walk at least two miles three times a week.

② **Converse** ($q \to p$)

If priya walks at ~~least~~ least two miles three times a week then she is concerned about her cholesterol levels.

③ Contrapositive: $(\sim q \to \sim p)$

If priya weathers does not walks at least two miles three times a week. then she is not concerned about her cholesterol levels.

④ inverse: $(\sim p \to \sim q)$

If priya is not concerned about her cholesterol levels then she will not walks

Ex: consider the statements.

p: you are guilty q: you are punished.

Ex: p: Today is a holiday.

q: I will go for a movie

Ex: let p, q and r be the propositions.

p: you have flu q: you miss the final examination

r: you pass the course.

Express each of the following formula as an English sentence.

a) $p \to q$ b) $\sim q \leftrightarrow r$ c) $q \to \sim r$ d) $p \vee q \vee r$.

e) $(p \to \sim r) \vee (q \to \sim r)$ f) $(p \wedge q) \vee (\sim q \vee r)$.

Ex: let p and q be the propositions:

p: I bought a lottery ticket this week.

q: I won the million dollar jackpot on friday.

Express each of the following formula as an English sentence:

a) $\sim p$

b) $p \to q$

c) $p \leftrightarrow q$

d) $\sim p \wedge \sim q$

e) $p \vee q$

f) $p \wedge q$

g) $\sim p \to \sim q$

h) $\sim p \vee (p \wedge q)$.

Functionally complete sets of connectives

① We have already defined the connectives $\wedge, \vee, \sim, \rightarrow$ and $\rightleftarrows$.

② Now introduce some other connectives namely NAND, NOR and XOR.

The word NAND is combination of NOT and AND

" NOR " " NOT and OR

" XOR " NOT and $\leftrightarrow$

$\therefore$ NAND = $\sim$ Conjunction $\sim\wedge$

NOR = $\sim$ disjunction $\sim\vee$

XOR = $\sim$ Biimplication. $\leftrightarrow$

③ The Connectives $\uparrow$ and $\downarrow$ have been defined in terms of the connectives $\wedge, \vee$ and $\sim$.

④ Therefore, for any formula containing the connectives $\uparrow$ or $\downarrow$, one can obtain an equivalent formula containing the connectives $\wedge, \vee$ and $\sim$ only.

⑤ Note:- $\uparrow$ and $\downarrow$ are duals of each other.

Therefore in order to obtain the dual of a formula which includes $\uparrow$ or $\downarrow$, we should interchange $\uparrow$ and $\downarrow$

Definition:- (functionally complete set of connectives)

if every formula can be expressed in terms of an equivalent formula containing the connectives only from this set.

(6) Functionally completed set should not contain any redundant connectives i.e

connective which can be expressed in terms of other connectives.

Already we have,

(1) $P \vee Q \Leftrightarrow \sim(\sim P \wedge \sim Q)$

(2) $P \wedge Q \Leftrightarrow \sim(\sim P \vee \sim Q)$.

(3) $P \to Q \Leftrightarrow \sim P \vee Q$.

(4) $P \leftrightarrow Q \Leftrightarrow (\sim P \vee Q) \wedge (P \vee \sim Q)$.

(7) Hence by first replacing all biconditionals, then the conditionals finally all the conjunctions or all the disjunctions in any formula.

(8) we can obtain an equivalent formula which contains either the negation and disjunction or only or the negation and conjunction only.

(9) In other words, for every formula, we can find an equivalent formula containing the connectives $\vee$ and $\sim$ or $\wedge$ and $\sim$ only.

(10) From the definition of functional complete set of connectives, the sets of connectives $\{\wedge, \sim\}$ and $\{\vee, \sim\}$ are functionally ~~comp~~ complete sets.

(11) The set $\{\wedge, \vee\}$ is not functionally completed, as for the formula $\sim p$, it is not possible to find an equivalent formula containing connectives only from the set $\{\wedge, \vee\}$.

(12) $\{\uparrow\}$, $\{\downarrow\}$ are functionally complete.

<u>Proof:</u> In order to prove, it is sufficient to show that the sets of connectives $\{\wedge, \sim\}$ and $\{\vee, \sim\}$ can be expressed either in terms of $\uparrow$ alone or in terms of $\downarrow$ alone.

To show that $\{\vee, \sim\}$ is functionally complete, it is enough to show that $\sim$ and $\vee$ can be expressed in terms of $\downarrow$ alone.

We have

$$\sim p \Leftrightarrow \sim p \wedge \sim p \Leftrightarrow \sim(p \vee p) \Leftrightarrow p \downarrow p.$$

$$p \vee Q \Leftrightarrow \sim(\sim p \wedge \sim Q) \Leftrightarrow \sim p \downarrow \sim Q \Leftrightarrow$$

$\sim(\sim p \vee \sim Q)$

$p \wedge Q.$

$\sim \sim(p \vee Q)$

$\sim[p \downarrow Q]$

$$(\sim p \wedge \sim p) \downarrow (\sim Q \wedge \sim Q) \Rightarrow$$

$$\sim(p \vee p) \downarrow \sim(Q \vee Q)$$

$$\Rightarrow (p \downarrow p) \downarrow (Q \downarrow Q).$$

Then $\{\downarrow\}$ is a functionally completed set.

(13) To show the $\{\sim, \wedge\}$ is functionally completed, we have to express $\sim$ and $\wedge$ in terms of $\uparrow$ alone.

The following valid equalities help us in this direction.

$\sim p \Leftrightarrow \sim p \vee \sim p \Leftrightarrow \sim(p \wedge p) \Leftrightarrow p \uparrow p$.

$p \wedge q \Leftrightarrow \sim(\sim(p \wedge q))$ $\quad \sim(p\wedge q) \uparrow \sim(p \wedge q)$

$\Leftrightarrow \underline{\sim(p \uparrow q)}$ $\quad (p \uparrow q) \wedge (p \uparrow q)$

$\Leftrightarrow [(p \wedge q) \uparrow (p \uparrow q)]$.

Then $\{\uparrow\}$ is a functionally complete set.

$(p \uparrow q) \uparrow (p \uparrow q)$

$\sim(p \wedge q) \uparrow \sim(p \wedge q)$

$\sim(\sim(p \wedge q) \wedge \sim(p \wedge q))$ $\quad \sim(p \wedge q)$

$\sim(p \wedge q) \vee p \wedge q$

$p \wedge q$

$p \vee q = \sim(\sim(p \vee q))$

$= \sim(p \downarrow q)$

$(p \downarrow q) \downarrow (p \downarrow q)$ $\quad \sim(p \vee q) \downarrow \sim(p \vee q)$

$= \sim[\sim(p \vee q) \vee \sim(p \vee q)]$

$(p \vee q) \wedge (p \vee q)$

$\underline{p \vee q}$

① Prove that $(P\to Q)\wedge(R\to Q) \Rightarrow (P\vee R)\to Q$.

② Show that $(\sim P\wedge(\sim q\wedge r))\vee(q\wedge r)\vee(p\wedge r) \Leftrightarrow r$.

③ Show that implications.

ⓐ $\sim Q\wedge(P\to Q)\Rightarrow P$

$\sim Q\wedge(P\to Q)\to P$ is a T.

$\sim Q\wedge[\sim P\vee Q)]\to P$.

$\sim[\sim Q\wedge (\sim P\vee Q)]\vee P$.

$[Q\vee\sim(\sim P\vee Q)]\vee P$

$[Q\vee(P\wedge\sim Q)]\vee P$.

$[(Q\vee P)\wedge(Q\vee\sim Q)]\vee P$.

$((P\vee Q)\wedge T)\vee P$

$P\vee Q\vee P \Leftrightarrow P\vee P\vee Q \Leftrightarrow P\vee Q$.

It is not a tautological implication.

ⓑ $(P\vee Q)\wedge(\sim Q)\Rightarrow Q$

$(P\vee Q)\wedge(\sim Q)\to Q$ is T

$\sim[(P\vee Q)\wedge(\sim Q)]\vee Q$.

$[\sim(P\vee Q)\vee\sim(\sim Q)]\vee Q$.

$(\sim(P\vee Q)\vee Q)\vee(Q)$.

$(\sim P\wedge\sim Q)\vee Q)\vee Q$.

$(\sim P\vee Q)\wedge(\sim Q\vee Q)\vee Q$

(b) $(P \vee Q) \wedge (\sim P) \Rightarrow Q$.

Sol: $((P \vee Q) \wedge (\sim P)) \to Q$. is T

$\sim((P \vee Q) \wedge (\sim P)) \vee Q$.

$(\sim(P \vee Q) \vee P) \vee Q$.

$((\sim P \wedge \sim Q) \vee P) \vee Q$.

$((\sim P \vee P) \wedge (\sim Q \vee P)) \vee Q$.

$(T \wedge (P \vee \sim Q)) \vee Q$.

$(P \vee \sim Q) \vee Q$.

$P \vee Q \vee \sim Q \Rightarrow P \vee T \Rightarrow T$.

(c) $(P \to Q) \Rightarrow P \to (P \wedge Q)$.

$(P \to Q) \to (P \to (P \wedge Q))$ is T.

$(\sim P \vee Q) \to (P \to (P \wedge Q))$

$(\sim P \vee Q) \to (\sim P \vee (P \wedge Q))$

$(\sim P \vee Q) \to ((\sim P \vee P) \wedge (\sim P \vee Q))$

$(\sim P \vee Q) \to (T \wedge (\sim P \vee Q))$

$(\sim P \vee Q) \to (\sim P \vee Q)$.

$\sim(\sim P \vee Q) \vee (\sim P \vee Q)$

$(\sim P \vee Q) \vee \sim(\sim P \vee Q)$

$P \vee \sim P$

$T$.

$\therefore$ L.H.S = R.H.S $(P \vee Q \Leftrightarrow Q \vee P)$

Example: Prove that NAND ($\uparrow$) and NOR ($\downarrow$) both are commutative but not associative.

Ans:-

(i) $P \uparrow Q \Leftrightarrow Q \uparrow P$ , $P \downarrow Q \Leftrightarrow Q \downarrow P$. (Commutative)

by using truth table

| P | Q | $P\uparrow Q$ | $Q \uparrow P$ |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |

| P | Q | $P\downarrow Q$ | $Q \downarrow P$ |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 |

The connectives $\downarrow$, $\uparrow$ are commutative.

without constructing truth table

(i) $P \uparrow Q \Leftrightarrow \sim(P \wedge Q)$

$\Leftrightarrow \sim P \vee \sim Q$. — L.H.S

$Q \uparrow P \Leftrightarrow \sim(Q \wedge P)$

$\Leftrightarrow \sim Q \vee \sim P$

$\Leftrightarrow \sim P \vee \sim Q$ — R.H.S.

$\therefore$ L.H.S = R.H.S. $[\because (P \uparrow Q) \Leftrightarrow (Q \uparrow P)]$

(ii) $P \downarrow Q \Leftrightarrow Q \downarrow P$

$P \downarrow Q \Leftrightarrow \sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$ — L.H.S

$Q \downarrow P \Leftrightarrow \sim(Q \vee P) \Leftrightarrow \sim Q \wedge \sim P$

$\Leftrightarrow \sim P \wedge \sim Q$. — R.H.S

$\therefore$ L.H.S = R.H.S $(P \downarrow Q \Leftrightarrow Q \downarrow P)$

(iii) The connectives ↑ and ↓ are not associative.

i.e $P \uparrow (Q \uparrow R) \not\Leftrightarrow (P \uparrow Q) \uparrow R$.

$P \downarrow (Q \downarrow R) \not\Leftrightarrow (P \downarrow Q) \downarrow R$.

L.H.S — $P \uparrow (Q \uparrow R) \Leftrightarrow P \uparrow \sim(Q \wedge R)$

$\Leftrightarrow \sim(P \wedge \sim(Q \wedge R))$.

$\Leftrightarrow \sim P \vee \sim(\sim(Q \wedge R))$

$\Leftrightarrow \sim P \vee (Q \wedge R)$

$\Leftrightarrow (\sim P \vee Q) \wedge (\sim P \vee R)$

R.H.S — $(P \uparrow Q) \uparrow R \Leftrightarrow \sim(P \wedge Q) \uparrow R$

$\Leftrightarrow \sim(\sim(P \wedge Q) \wedge R)$.

$\Leftrightarrow (P \wedge Q) \vee (\sim R)$

$\Leftrightarrow (P \vee \sim R) \wedge (Q \vee \sim R)$.

$\therefore P \uparrow (Q \uparrow R) \not\Leftrightarrow (P \uparrow Q) \uparrow R$.

Similarly $P \downarrow (Q \downarrow R) \not\Leftrightarrow (P \downarrow Q) \downarrow R$.

By using truth table

| P | Q | R | $P \uparrow Q$ | $(P \uparrow Q) \uparrow R$ ① | $Q \uparrow R$ | $P \uparrow (Q \uparrow R)$ ② |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 | 1 | 0 | 1 |

both are not equal. Similarly $(P \downarrow Q) \downarrow R \not\Leftrightarrow P \downarrow (Q \downarrow R)$.

(d) $(p \to q) \to q \Rightarrow (p \vee q)$

$(p \to q) \to q \to (p \vee q)$

$[\sim(p \to q) \vee q] \to (p \vee q)$

$[\sim(\sim p \vee q) \vee q] \to (p \vee q)$

$\sim[\sim(\sim p \vee q) \vee q] \vee (p \vee q)$

$[(\sim p \vee q) \wedge \sim q] \vee (p \vee q)$

$[(\sim p \wedge \sim q) \vee (q \wedge \sim q)] \vee (p \vee q)$

$[(\sim p \wedge \sim q) \vee F] \vee (p \vee q)$

$(\sim p \wedge \sim q) \vee (p \vee q)$

$(\sim p \vee p \vee q) \wedge (\sim q \vee p \vee q)$

$(T \vee q) \wedge (T \vee p) \Rightarrow p \wedge q.$