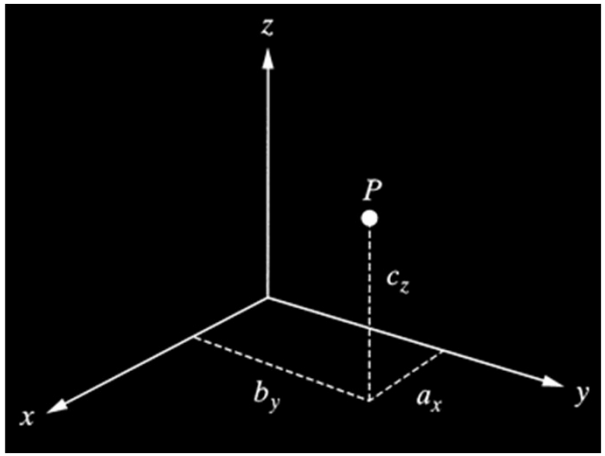
Robot Kinematics: Position Analysis

- Forward Kinematics:
  to determine where the robot's hand is?
  (If all joint variables are known)

- Inverse Kinematics:
  to calculate what each joint variable is?
  (If we desire that the hand be located at a particular point)

Representation Of A Point In Space

- A point P in space :
  - 3 coordinates relative to a reference frame

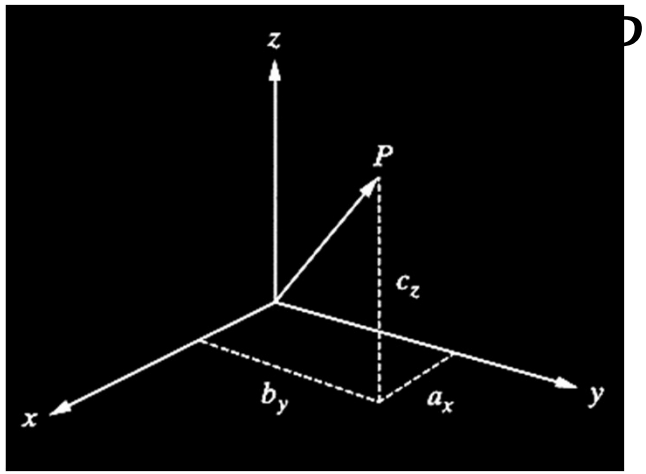


Representation of a point in space

$$P = a_x \hat{i} + b_y \hat{j} + c_z \hat{k}$$

Representation of a Vector in Space

- A Vector P in space :
  3 coordinates of its tail and of its head



Representation of a vector in space

$$P = a_x i + b_y j + c_z k$$

$$\bar{P} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

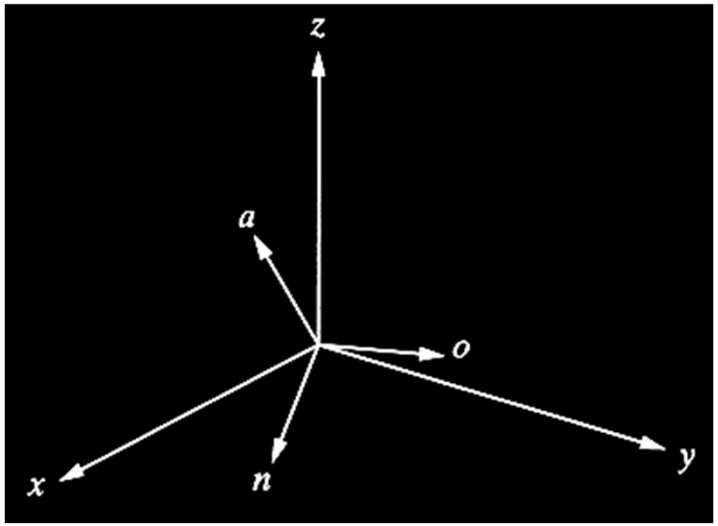
Where w is Scale factor

Scale Factor

- It can Change overall size of the vector similar to zooming function in computer graphics.
  - When w =1 ,
    - size of components remain unchanged
  - When w =0,
    - It represent a vector whose length is infinite but it represents the direction so called as directional vector

Representation of a Frame at the Origin of a Fixed-Reference Frame

- Each Unit Vector is mutually perpendicular. :
  normal, orientation, approach vector

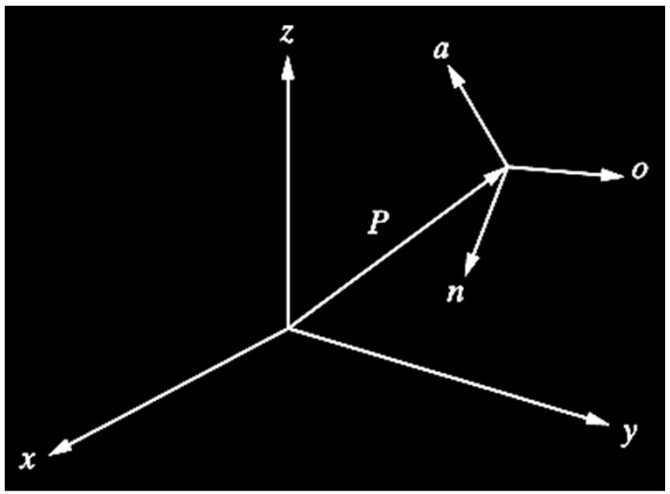


$$F = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}$$

Representation of a frame at the origin of the reference frame

Representation of a Frame in a Fixed Reference Frame

- Each Unit Vector is mutually perpendicular. : normal, orientation, approach vector

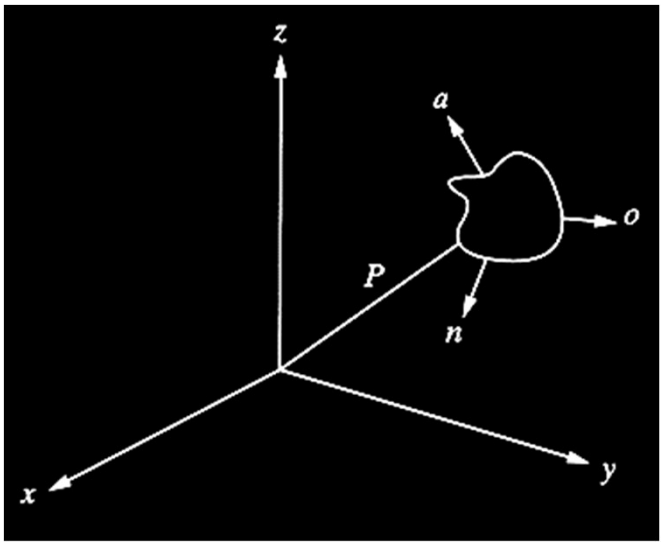


$$F = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Representation of a frame in a Fixed Reference Frame

Representation of a Rigid Body

- An object can be represented in space by attaching a frame to it and representing the frame in space.



$$F_{object} = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Representation of an object in space

Example problem on frame definition

- For the following frame, find the values of the missing elements and complete the matrix representation of the frame:

$$F = \begin{bmatrix} ? & 0 & 0 & 5 \\ 0.707 & ? & ? & 3 \\ ? & ? & ? & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: Obviously, the 5,3,2 values representing the position of the origin of th frame do not affect the constraint equations. Please notice that only 3 values fo directional vectors are given. This is all that is needed.

$$n_x o_x + n_y o_y + n_z o_z = 0 \quad \text{or} \quad n_x(0) + 0.707(o_y) + n_z(o_z) = 0$$

$$n_x a_x + n_y a_y + n_z a_z = 0 \quad \text{or} \quad n_x(a_x) + 0.707(a_y) + n_z(0) = 0$$

$$a_x o_x + a_y o_y + a_z o_z = 0 \quad \text{or} \quad a_x(0) + a_y(o_y) + 0(o_z) = 0$$

$$n_x^2 + n_y^2 + n_z^2 = 1 \quad \text{or} \quad n_x^2 + 0.707^2 + n_z^2 = 1$$

$$o_x^2 + o_y^2 + o_z^2 = 1 \quad \text{or} \quad 0^2 + o_y^2 + o_z^2 = 1$$

$$a_x^2 + a_y^2 + a_z^2 = 1 \quad \text{or} \quad a_x^2 + a_y^2 + 0^2 = 1$$

Simplifying these equations yields:

$$0.707\, o_y + n_z o_z = 0$$

$$n_x a_x + 0.707\, a_y = 0$$

$$a_y o_y = 0$$

$$n_x^2 + n_z^2 = 0.5$$

$$o_y^2 + o_z^2 = 1$$

$$a_x^2 + a_y^2 = 1$$

Solving these six equations will yield $n_x = \pm 0.707$, $n_z = 0$, $o_y = 0$, $o_z = 1$, $a_x = \pm 0.707$, and $a_y = -0.707$. Notice that both n_x and a_x must have the same sign. The reason for multiple solutions is that with the given parameters, it is possible to have two sets of mutually perpendicular vectors in opposite directions. The final matrix will be:

$$F_1 = \begin{bmatrix} 0.707 & 0 & 0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} -0.707 & 0 & -0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the missing elements of the following frame representation:

$$F = \begin{bmatrix} ? & 0 & ? & 3 \\ 0.5 & ? & ? & 9 \\ 0 & ? & ? & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution:

$$n_x^2 + n_y^2 + n_z^2 = 1 \quad \rightarrow \quad n_x^2 + 0.25 = 1 \quad \rightarrow \quad n_x = 0.866$$

$$\mathbf{n} \cdot \mathbf{o} = 0 \quad \rightarrow \quad (0.866)(0) + (0.5)(o_y) + (0)(o_z) = 0 \quad \rightarrow \quad o_y = 0$$

$$|\mathbf{o}| = 1 \quad \rightarrow \quad o_z = 1$$

$$\mathbf{n} \times \mathbf{o} = \mathbf{a} \quad \rightarrow \quad \mathbf{i}(0.5) - \mathbf{j}(0.866) + \mathbf{k}(0) = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$a_x = 0.5$$

$$a_y = -0.866$$

$$a_z = 0$$

Homogeneous Transformation Matrices

- A transformation matrices must be in square form.
- It is much easier to calculate the inverse of square matrices.
- To multiply two matrices, their dimensions must match.

$$F = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformations

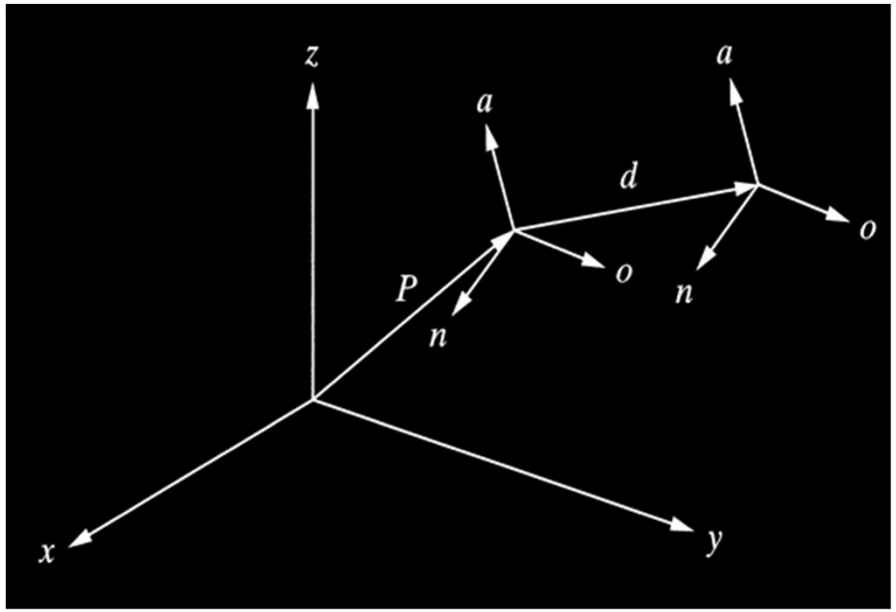
☞ A transformation is defined as making a movement in space.

➢ Types of Transformation are:

- A pure translation
- A pure rotation
- A combination of translation and rotation

Representation of a Pure Translation

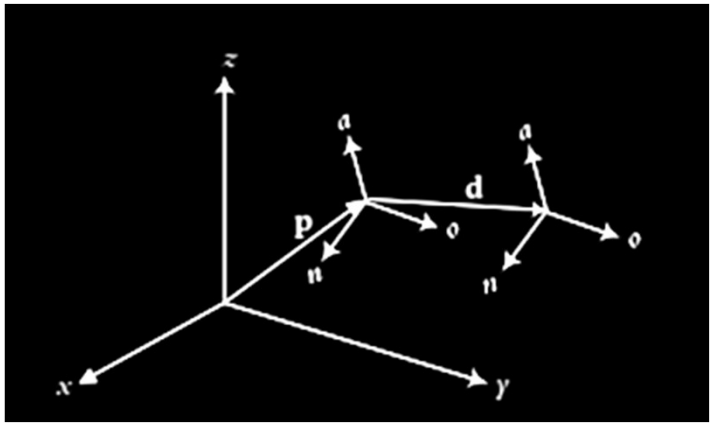
- If a frame moves in space without any change in its orientation
- ✓ Assumption : The frame is at the origin of the reference frame and parallel to it.



$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Representation of an pure translation in space

A pure translation



$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{x} \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & P_x + d_x \\ n_y & o_y & a_y & P_y + d_y \\ n_z & o_z & a_z & P_z + d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The equation is also symbolically written as :

$$F_{new} = Trans(d_x \quad d_y \quad d_z) \text{ x } F_{old}$$

Numerical Problem-1

For Vector $v = 25i + 10j + 20k$, perform a translation by a distance of 8 in the x-direction, 5 in the y-direction, and 0 in the z-direction.

Answer:

$$v_{new} = \text{Trans}(d_x, d_y, d_z) \times v_{old} = Trans(8,5,0) \times v_{old}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 25 \\ 10 \\ 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 33 \\ 15 \\ 20 \\ 1 \end{bmatrix}$$

Numerical Problem-2

A frame F has been moved 10 units along y-axis and 5 units along z-axis of reference frame. Find new location of frame.

$$F = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer:

$$F_{new} = \mathrm{Trans}(\mathrm{d_x}, \mathrm{d_y}, \mathrm{d_z}) \times F_{old} = Trans(0,10,5) \times F_{old}$$

Numerical Problem-2

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 13 \\ -0.766 & 0 & 0.643 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Pure Rotation about an Axis

- Assumption : The frame is at the origin of the reference frame and parallel to it.

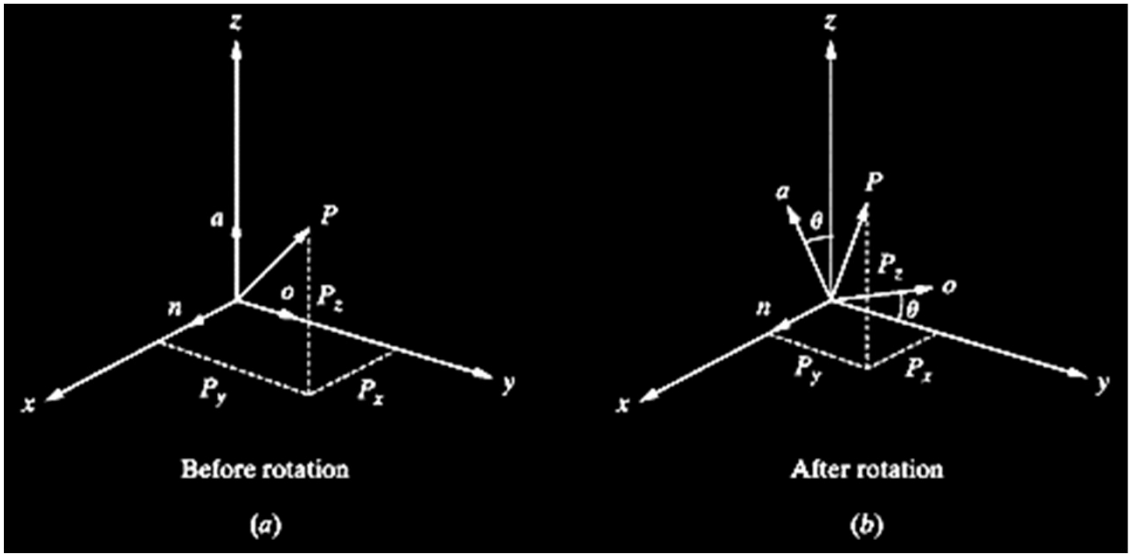


Fig. 2.10 Coordinates of a point in a rotating frame before and after rotation.

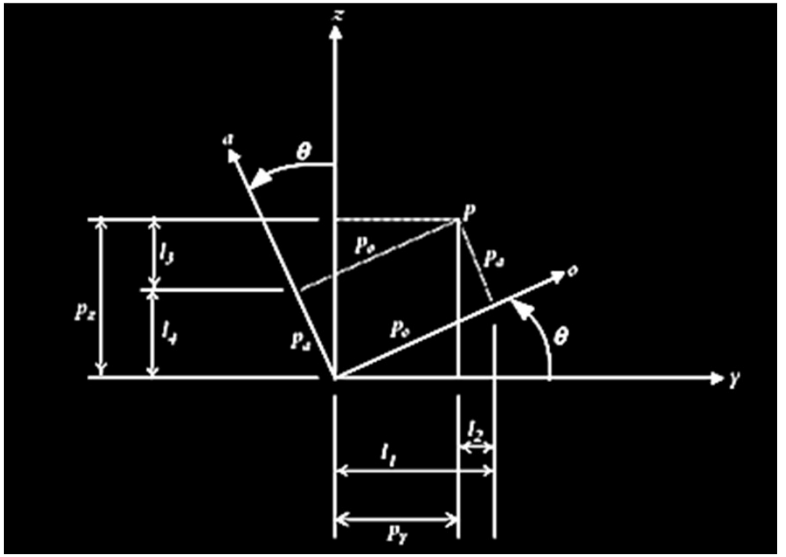
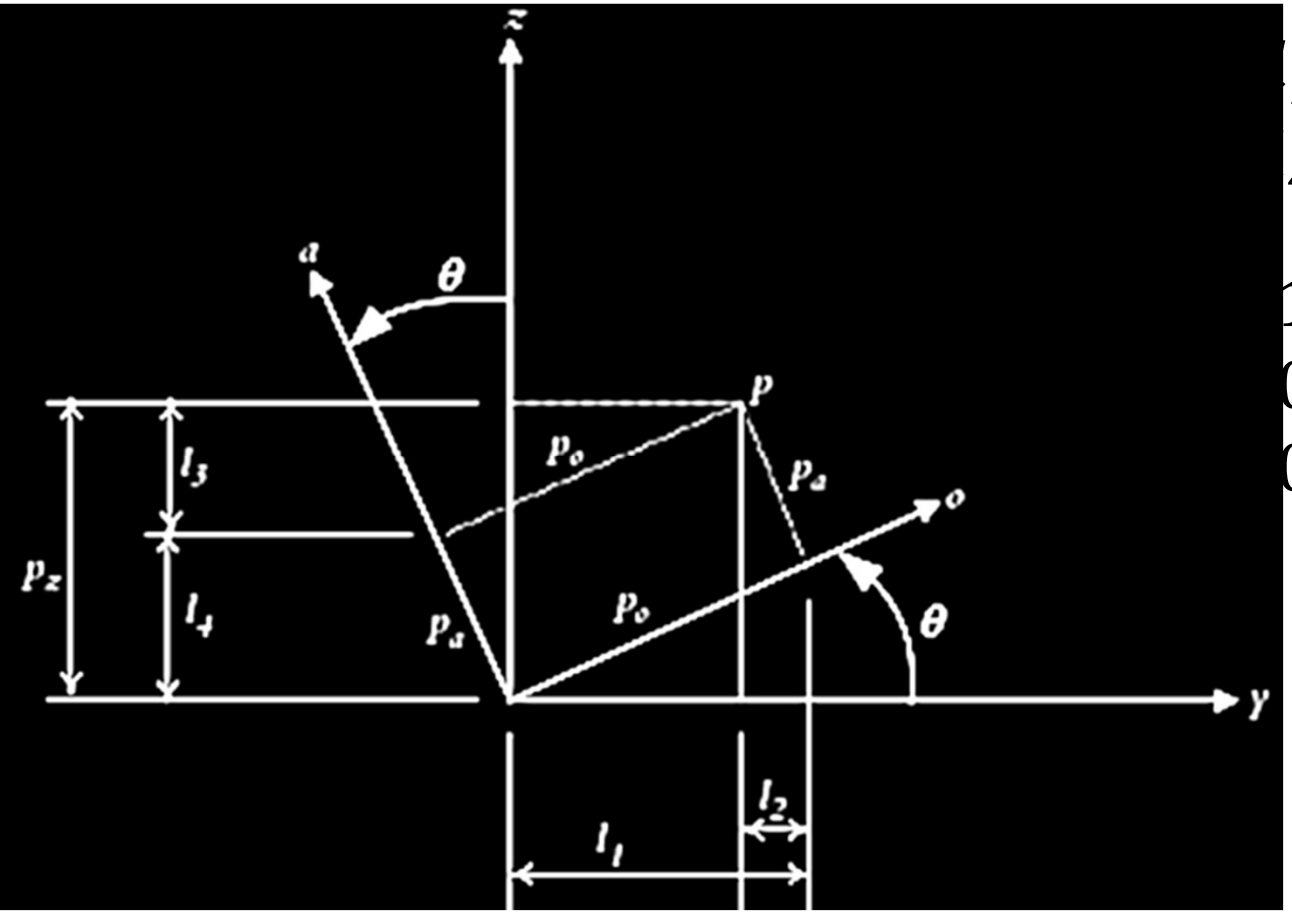


Fig. 2.11 Coordinates of a point relative to the reference frame and rotating frame as viewed from the *x*-axis.



$$p_x = p_n$$

$$_2 = p_o \cos\theta - p_a \sin\theta$$

$$_4 = p_o \sin\theta + p_a \cos\theta$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_n \\ p_o \\ p_a \end{bmatrix}$$

Pure Rotation about an Axis

$$Rot(z,\theta) = \left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Numerical Problem-3

Rotate Vector $v = 5i + 3j + 8k$, by an angle of 90^o about the x-axis.

Answer:

$$v_{new} = \text{Rot}(\text{x}, 90^o) \times v_{old} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ 3 \end{bmatrix}$$

Combined Transformations

- Combined Transformation consist of a number of successive translations and rotations about fixed reference frame axes.
- As you can see, the coordinates of the point relative to the reference frame at the conclusion of each transformation is found by pre-multiplying the coordinates of the point by each transformation matrix.
- The order of matrices written is the opposite of the order of transformations performed.
- If the order of matrices changes then final position of robot also changes

Numerical Problem (Combined transformations)

☛A point p(7,3,1) is attached to frame and subjected to following transformations. Find coordinate of point relative to reference frame.

1. Rotation of 90° about z-axis
2. Followed by rotation of 90 about y-axis
3. Followed by translation of [4,-3,7].

Answer: The matrix equation is given as

$$p_{xyz} = Trans(4, -3, 7)Rot(y, 90)Rot(z, 90)p_{noa}$$

Numerical Problem

$$p_{xyz} = Trans(4, -3, 7)Rot(y, 90)Rot(z, 90)p_{noa}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 10 \\ 1 \end{bmatrix}$$

Numerical Problem-4

For Vector $v = 2i + 5j + 3k$, is rotated by 60^o about z-axis and translated by 3, 4 and 5 units in the x, y and z directions. Find the vector with reference to reference frame

Answer:

$$v_{new} = \text{Trans}(\text{d}_\text{x}, \text{d}_\text{y}, \text{d}_\text{z}) \times Rot(z, \theta) \times v_{old}$$

$$= Trans(3,4,5) \times Rot(z, 60^o) \times v_{old}$$

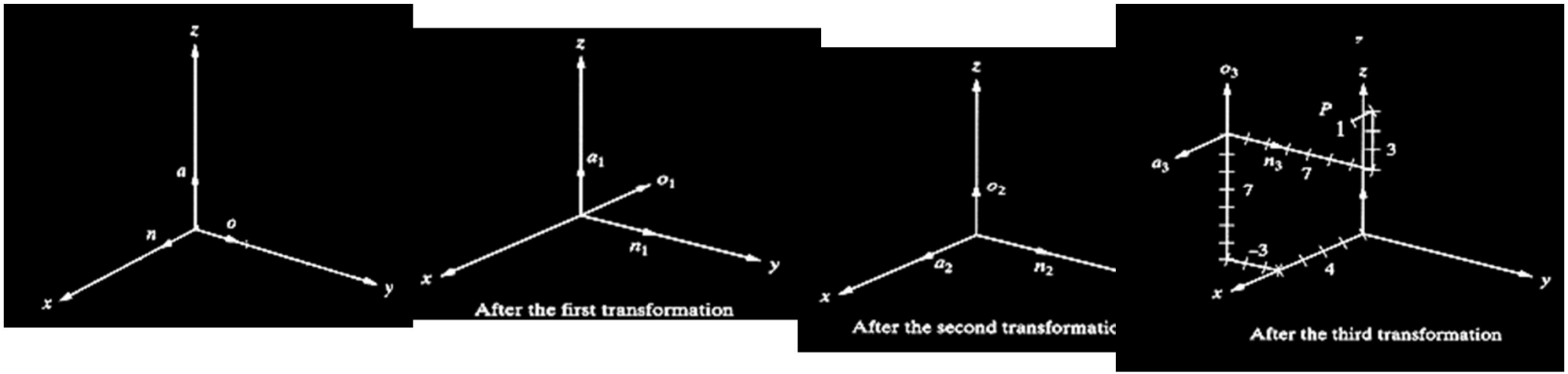
$$= \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & -0.866 & 0 & 0 \\ 0.866 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.33 \\ 8.232 \\ 8 \\ 1 \end{bmatrix}$$

Numerical Problem- Graphical Method

☛A point p(7,3,1) is attached to frame and subjected to following transformations. Find coordinate of point relative to reference frame.

1. Rotation of 90° about z-axis
2. Followed by rotation of 90 about y-axis
3. Followed by translation of [4,-3,7].



Numerical Problem -5(Combined transformations)

☛A point p(7,3,1) is attached to frame and subjected to following transformations. Find coordinate of point relative to reference frame.

1. Rotation of 90° about z-axis
2. Followed by translation of [4,-3,7]
3. Followed by rotation of 90° about y-axis

Answer: The matrix equation is given as

$$p_{xyz} = Rot(y, 90)\,Trans(4,-3,7)\,Rot(z, 90)\,p_{noa}$$

Numerical Problem -2(Combined transformations)

$$p_{xyz} = Rot(y, 90)\, Trans(4, -3, 7)\, Rot(z, 90)\, p_{noa}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

Transformations Relative to the Rotating Frame

- All transformations we have discussed so far have been relative to the fixed reference frame. This means that all translations, rotations, and distances (except for the location of a point relative to the moving frame) have been measured relative to the reference frame axes.
- However, it is possible to make transformations relative to the axes of a moving or current frame.
- This means that, for example, a rotation of 90^o may be made relative to the n-axis of the moving frame (also referred to as the current frame), and not the x-axis of the reference frame.

Transformations Relative to the Rotating Frame

- To calculate the changes in the coordinates of a point attached to the current frame relative to the reference frame, the transformation matrix is post-multiplied instead.
- Note that since the position of a point or an object attached to a moving frame is always measured relative to that moving frame, the position matrix describing the point or object is also always post-multiplied.

Numerical Problem (Combined transformations)

☛A point p(7,3,1) is attached to frame and subjected to following transformations. Find coordinate of point relative to reference frame.

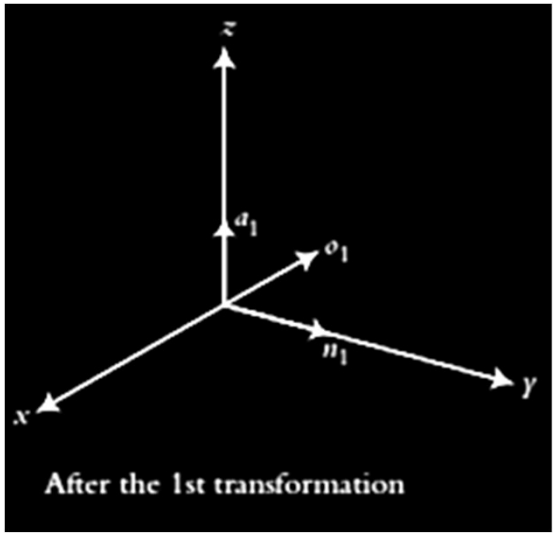
1. Rotation of 90° about a-axis
2. Followed by translation of [4,-3,7] along n-,o-,a- axes
3. Followed by rotation of 90 about o-axis.

Answer: The matrix equation is given as

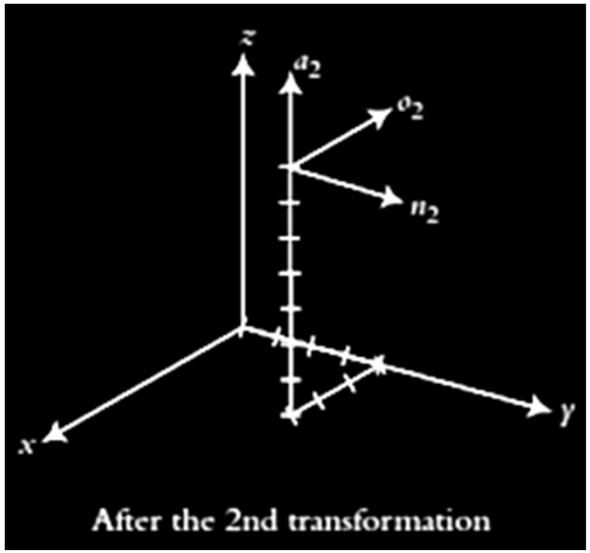
$$p_{xyz} = Rot(a, 90)\,Trans(4, -3, 7)\,Rot(o, 90)\,p_{noa}$$

$$p_{xyz} = Rot(a, 90)\,Trans(4, -3, 7)\,Rot(o, 90)\,p_{noa}$$

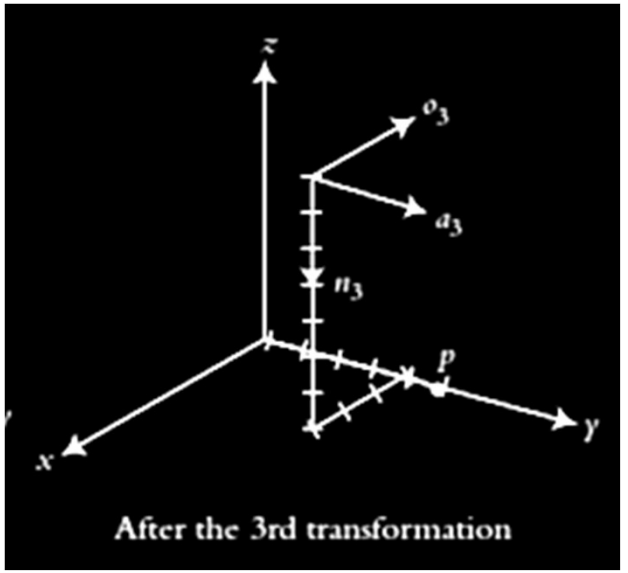
$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$



After the 1st transformation



After the 2nd transformation



After the 3rd transformation

Numerical problem 3

A frame B was rotated about the x-axis 90^o then it was translated about the current a-axis 3 units before it was rotated about the z-axis 90^o. Finally, it was translated about current o–axis 5 units.

a) Write an equation that describes the motions.
b) Find the final location of a point $p(1,5,4)^T$ attached to the frame relative to the reference frame.

Answer: In this case, motions alternate relative to the reference frame and current frame.

(a) Pre- or post-multiplying each motion's matrix accordingly, we will get:

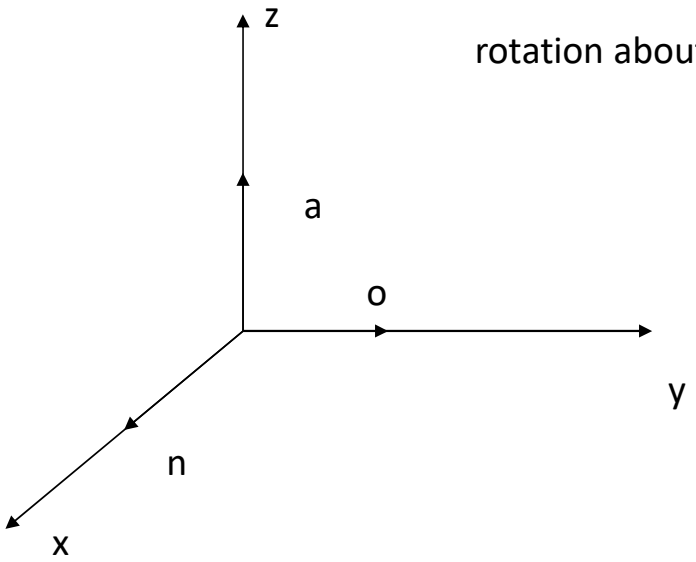
$$^{U}T_{B} = Rot(z,90)Rot(x,90)Trans(0,0,3)Trans(0,5,0)$$

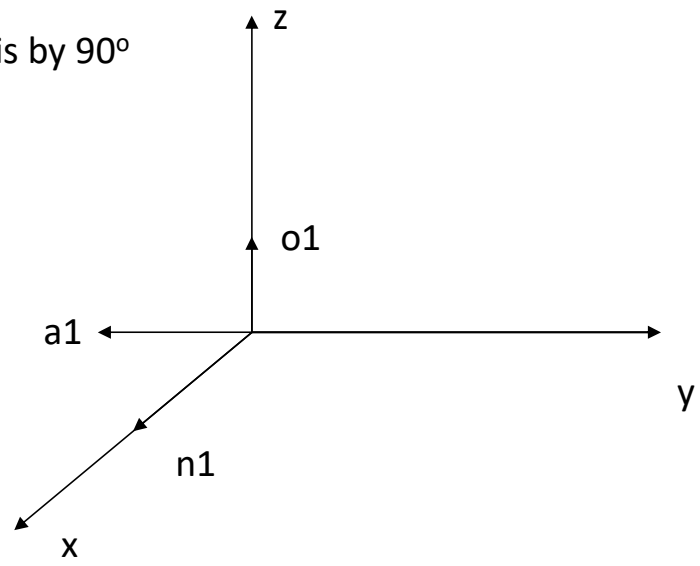
$$^{U}T_{B} = Rot(z,90)Rot(x,90)Trans(0,0,3)Trans(0,5,0)$$

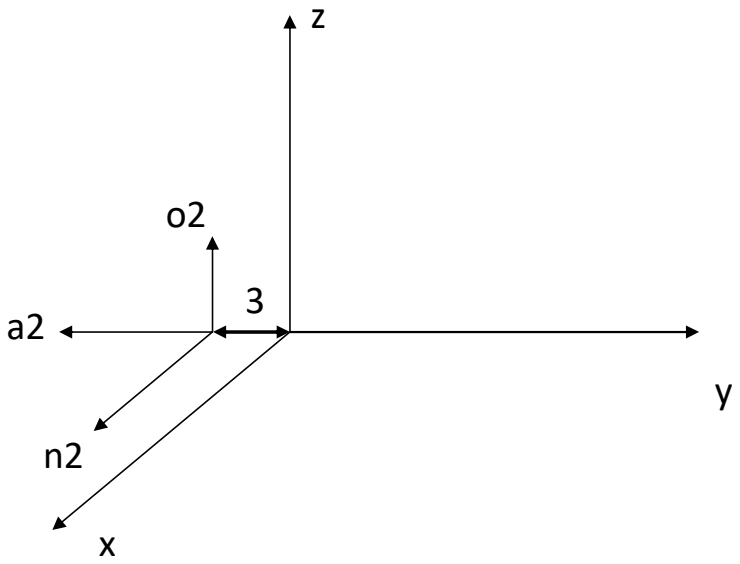
$$^{U}p = {}^{U}T_{B} \times {}^{B}p$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 10 \\ 1 \end{bmatrix}$$

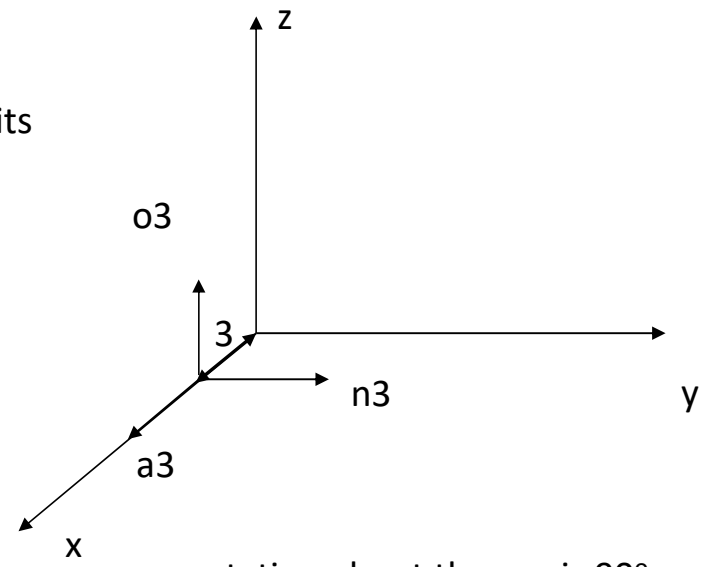
rotation about the x-axis by 90°



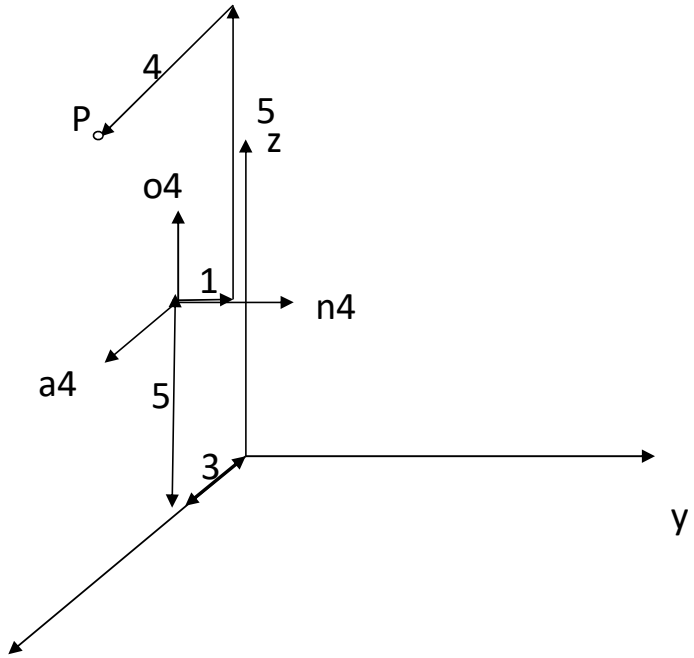




Translation about the current a-axis 3 units



rotation about the z-axis 90^{o}

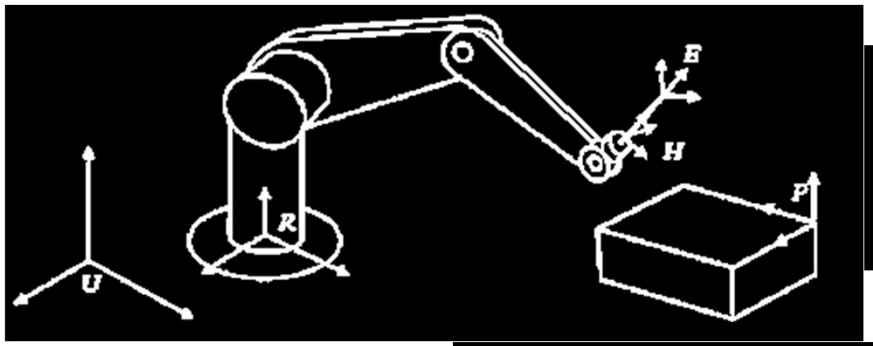


translation about current o–axis 5 units.

a point $p(1,5,4)^T$ attached to the frame

Inverse of Transformation Matrices

- There are many situations where the inverse of a matrix will be needed in robotic analysis. One situation where transformation matrices may be involved can be seen in the following example.



$$ {}^{U}T_{E} = {}^{U}T_{R}\,{}^{R}T_{H}\,{}^{H}T_{E} = {}^{U}T_{P}\,{}^{P}T_{E} $$

$$ ({}^{U}T_{R})^{-1}({}^{U}T_{R}\,{}^{R}T_{H}\,{}^{H}T_{E})({}^{H}T_{E})^{-1} = ({}^{U}T_{R})^{-1}({}^{U}T_{P}\,{}^{P}T_{E})({}^{H}T_{E})^{-1} $$

$$ {}^{R}T_{H} = {}^{U}T_{R}^{-1}\,{}^{U}T_{P}\,{}^{P}T_{E}\,{}^{H}T_{E}^{-1} $$

Inverse of Transformation Matrices

$$T = \begin{bmatrix} \overbrace{\begin{matrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{matrix}}^{\mathbf{R}} & \overbrace{\begin{matrix} P_x \\ P_y \\ P_z \end{matrix}}^{\mathrm{P}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \qquad T = \left[\begin{array}{ccc|c} & R_{3x3} & & P_{3x1} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$T^{-1} = \left[\begin{array}{ccc|c} & R_{3x3}^{T} & & \begin{matrix} -p.n \\ -p.o \\ -p.a \end{matrix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad T^{-1} = \left[\begin{array}{ccc|c} n_x & n_y & n_z & -p.n \\ o_x & o_y & o_z & -p.o \\ a_x & a_y & a_z & -p.a \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Numerical problem 4

- Calculate the inverse of the given transformation matrix:

$$T = \begin{bmatrix} 0.5 & 0 & 0.866 & 3 \\ 0.866 & 0 & -0.5 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer:

$$T^{-1} = \begin{bmatrix} 0.5 & 0.866 & 0 & -(3 \times 0.5 + 2 \times 0.866 + 5 \times 0) \\ 0 & 0 & 1 & -(3 \times 0 + 2 \times 0 + 5 \times 1) \\ 0.866 & -0.5 & 0 & -(3 \times 0.866 + 2 \times -0.5 + 5 \times 0) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.866 & 0 & -3.23 \\ 0 & 0 & 1 & -5 \\ 0.866 & -0.5 & 0 & -1.598 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward and Inverse Kinematics Equations for Position

Forward Kinematics and Inverse Kinematics equation for position analysis and three types of standard robot coordinate system are:

(a)Cartesian (gantry, rectangular) coordinates.
(b)Cylindrical coordinates.
(c)Spherical coordinates.

$$ {}^{R}T_{P} = T_{cart} = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

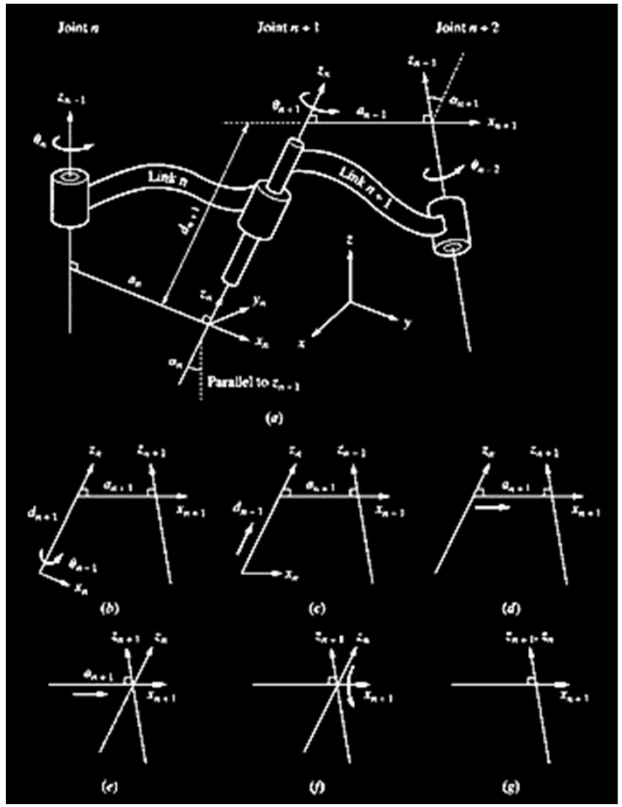
DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

- Denavit-Hartenberg Representation :

@ Simple way of modeling robot links and joints for any robot configuration, regardless of its sequence or complexity.

@ Transformations in any coordinates is possible.

@ Any possible combinations of joints and links and all-revolute articulated robots can be represented.



DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

- Procedures for assigning a local reference frame to each joint:

- All joints are represented by a *z*-axis.

  (right-hand rule for rotational joint, linear movement for prismatic joint)
- The common normal is one line mutually perpendicular to any two skew lines.
- Parallel *z*-axes joints make a infinite number of common normal.
- Intersecting *z*-axes of two successive joints make no common normal between them(Length is 0.).

DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

- Symbol Terminologies :

  θ : A rotation about the *z*-axis.

  d : The distance on the *z*-axis.

  a : The length of each common normal (Joint offset).

  α : The angle between two successive *z*-axes (Joint twist)

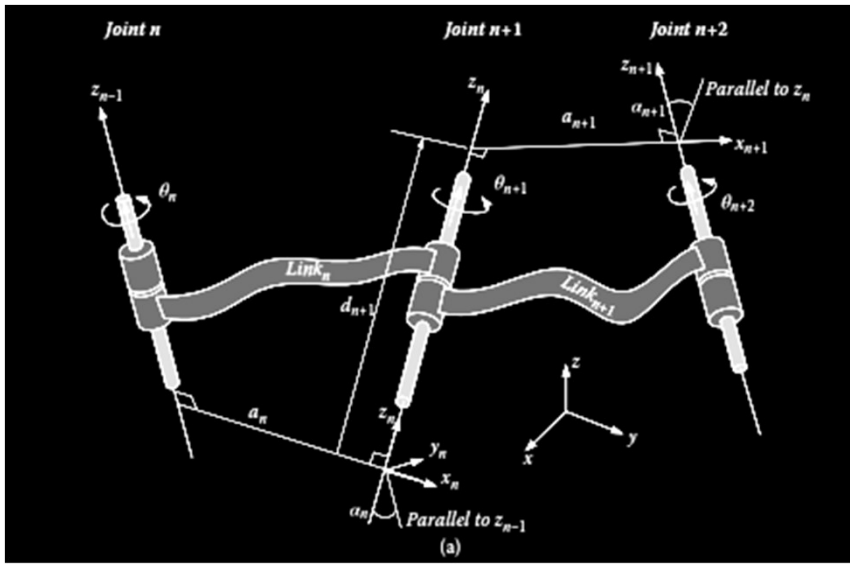
  ✋ **Only** θ and *d* are joint variables.

DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

- Robots may be made of a succession of joints and links in any order.
- The joints may be either prismatic (linear) or revolute (rotational), move in different planes, and have offsets.
- The links may also be of any length, including zero, may be twisted and bent, and may be in any plane.
- We need to be able to model and analyse any robot configuration, whether or not it follows any of the specific coordinates.
- To do this, we assign a reference frame to each joint and use a general procedure to transform from one joint to the next (one frame to the next).
- If we combine all the transformations from the base to the first joint, from the first joint to the second joint, etc., until we get to the last joint, we will have the robot's total transformation matrix.

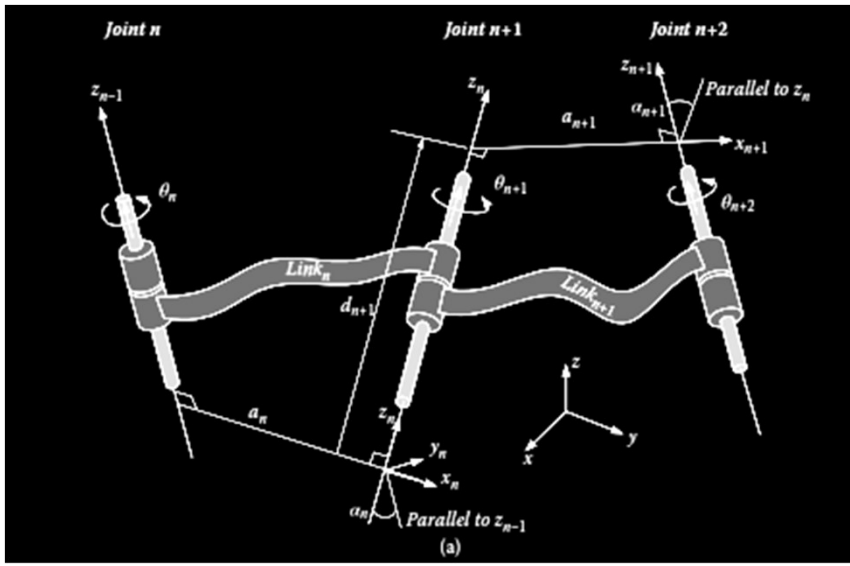
DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

- A robot may be made of a series of links and joints in any form.
- Figure represents three successive joints and two links between them. Although these joints and links are not necessarily similar to any real robot joint or link, they are very general and can easily represent any joint or link in real robots.
- The joints may be revolute or prismatic, or both.
- Although in real robots it is customary to only have 1-DOF joints, the joints in Figure represent 1- or 2-DOF joints.



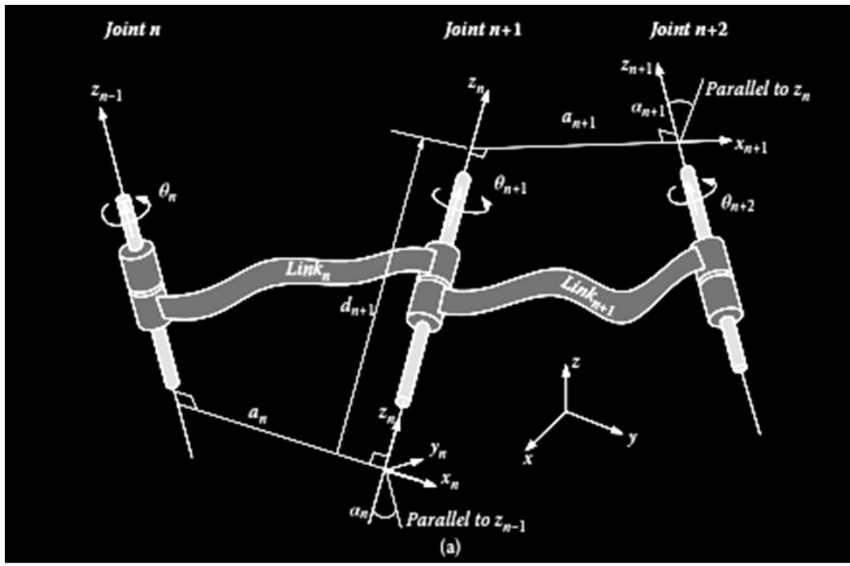
DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

- we assign joint number n to the first joint, n+1 to the second joint, and n+2 to the third joint.
- There may be other joints before or after these.
- Each link is also assigned a link number as shown.
- Link n will be between joints n and n + 1, and link n +1 is between joints n +1 and n +2.
- Although this seems tedious, in practice it is very intuitive and simple.



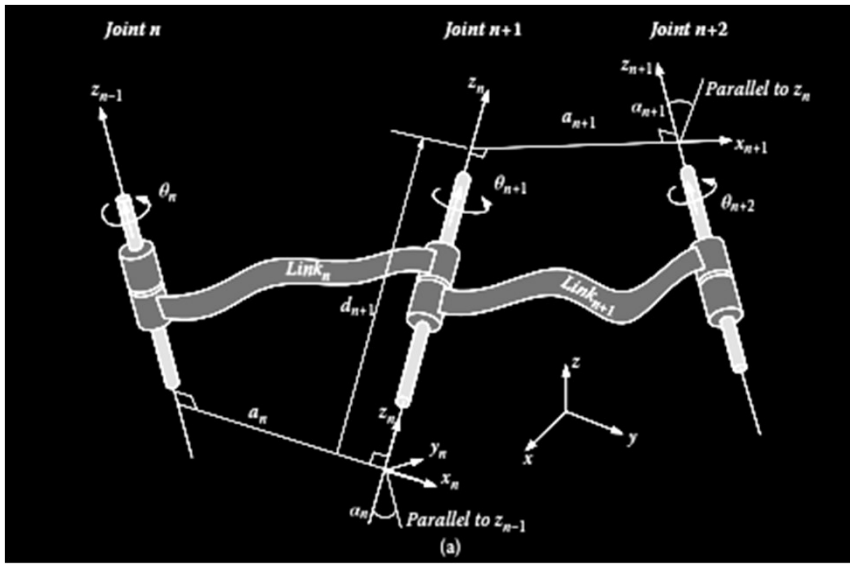
DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

- To model the robot with the D-H representation, we first assign a local reference frame to each and every joint.
- To each joint, we assign a local z-axis and an x-axis (although for moving frames, we have used the n-, o-,and a- designations, for simplicity we use x, y, and z here).
- We normally do not need to assign a y-axis, since we always know that y-axes are mutually perpendicular to both x- and z-axes. In addition, the Denavit Hartenberg representation does not use the y-axis at all.



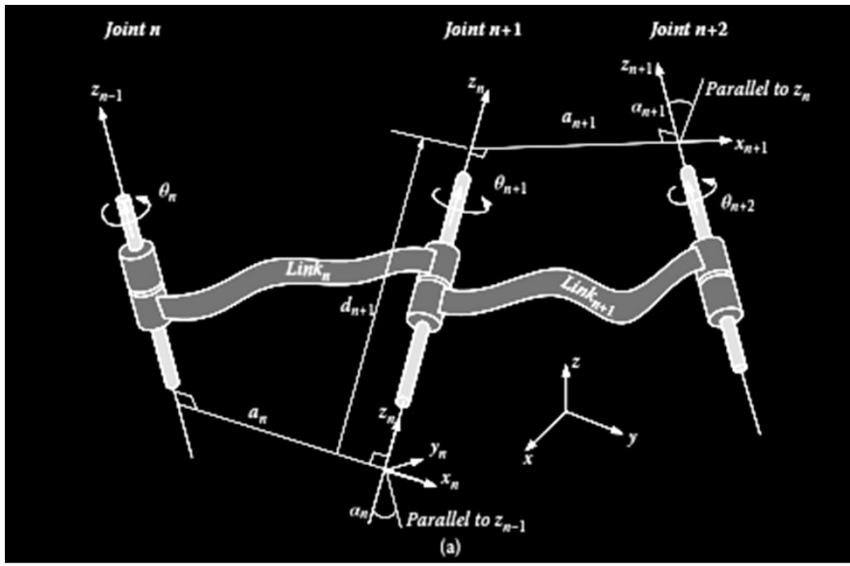
DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

- All joints, without exception, are represented by a z-axis.
- If the joint is revolute, the z-axis is in the direction of rotation as followed by the right-hand rule for rotations
- If the joint is prismatic, the z-axis is along the direction of the linear movement.
- In each case, the index number for the z-axis of joint n is n – 1. For example, the z-axis representing motions about joint number n+1is z_n.
- These simple rules allow us to quickly assign z-axes to all joints.
- For revolute joints, the rotation about the z-axis represented by (θ) is the joint variable.
- For prismatic joints, the length of the link along the z-axis represented by d is the joint variable.



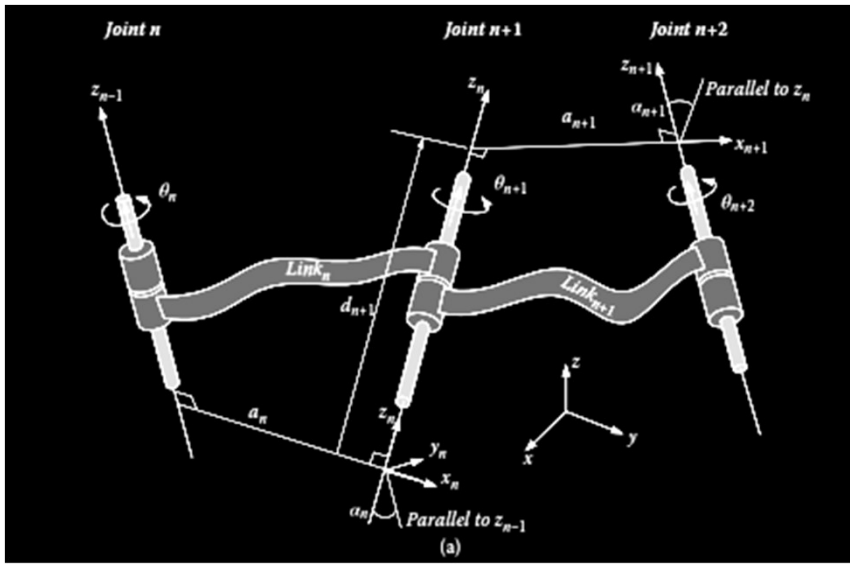
DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

- As shown in Figure, in general, joints may not necessarily be parallel or intersecting. As a result, the z-axes may be skew lines.
- There is always one line mutually perpendicular to any two skew lines, called the common normal, which is the shortest distance between them.
- We always assign the x-axis of the local reference frame in the direction of the common normal between the previous and current axes.
- Therefore, if a_n represents the common normal between z_{n-1} and z_n, the direction of x_n is along a_n.
- Similarly, if the common normal between z_n and z_{n+1} is a_{n+1}, the direction of x_{n+1} will be along a_{n+1}.



DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

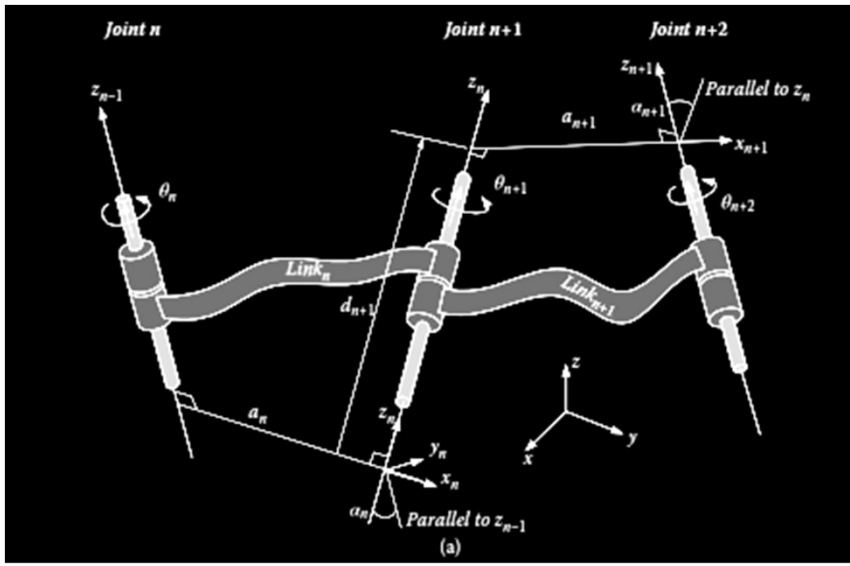
- The common normal lines between successive joints are not necessarily intersecting or collinear. As a result, the origins of two successive frames may also not be at the same location. Based on the above, we can assign coordinate frames to all joints, with the following special cases:



  - If two z-axes are parallel, there are an infinite number of common normals between them. We pick the common normal that is collinear with the common normal of the previous joint. This will simplify the model.
  - If the z-axes of two successive joints are intersecting, there is no common normal between them (or it has a zero length). We assign the x-axis along a line perpendicular to the plane formed by the two axes. This means that the common normal is a line perpendicular to the plane containing the two z-axes, which is equivalent of picking the direction of the cross-product of the two z-axes. This also simplifies the model.

DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

- In Figure, θ represents a rotation about the z-axis,
- d represents the distance on the z-axis between two successive common normal (or joint offset),
- a represents the length of each common normal (the length of a link), and
- α represents the angle between two successive z-axes (also called the twist angle).
- Commonly, only θ and d are joint variables. Values a and α are known physical parameters.



DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

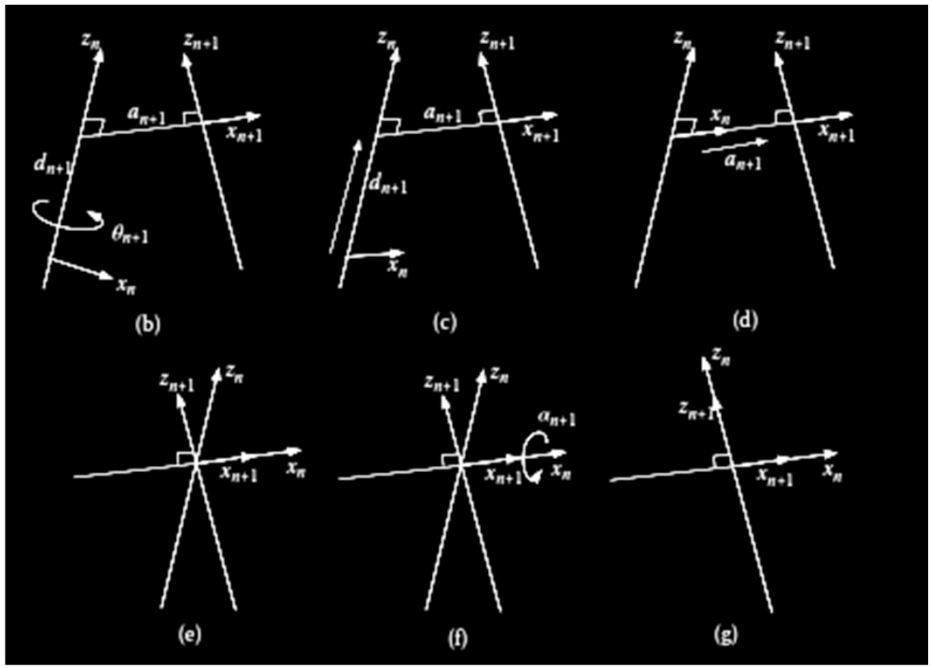
- The necessary motions to transform from one reference frame to the next.

(I) Rotate about the z_n-axis an angle of θ_{n+1}. This will make x_n parallel to x_{n+1}. (Coplanar)

(II) Translate along z_n-axis a distance of d_{n+1} to make x_n and x_{n+1} colinear.

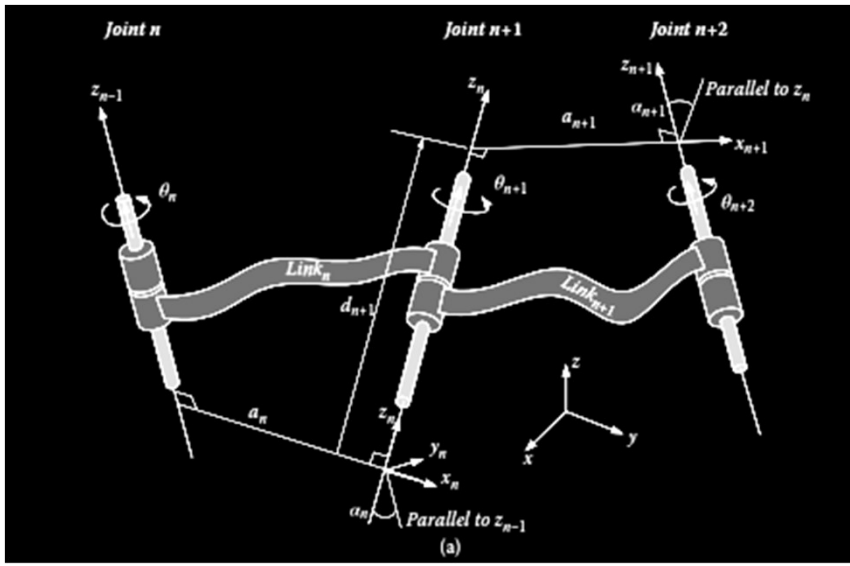
(III) Translate along the x_n-axis a distance of a_{n+1} to bring the origins of x_n and x_{n+1} together.

(IV) Rotate z_n-axis about x_{n+1} axis an angle of α_{n+1} to align z_n-axis with z_{n+1}-axis.



DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

- Repeating the exact same four motions between the successive next frames n+1 and n+2, etc., results in the total transformation between the reference frame and the end frame.
- Starting with the robot's reference frame, we can transform to the first joint, second joint, etc., until the end effector.
- Note that this sequence of movements remains the same between any two frames.



DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

- The transformation called A_{n+1} between two successive frames representing the four movements is the product of the four matrices representing them.
- Since all transformations are relative to the current frame (they are measured and performed relative to the axes of the current local frame) all matrices are postmultiplied.
- The result is:

$$A_{n+1} = Rot(z,\theta_{n+1}) \times Trans(0,0,d_{n+1}) \times Trans(a_{n+1},0,0) \times Rot(x,\alpha_{n+1})$$

$$= \begin{bmatrix} C\theta_{n+1} & -S\theta_{n+1} & 0 & 0 \\ S\theta_{n+1} & C\theta_{n+1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_{n+1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & a_{n+1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_{n+1} & -S\alpha_{n+1} & 0 \\ 0 & S\alpha_{n+1} & C\alpha_{n+1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta_{n+1} & -S\theta_{n+1}C\alpha_{n+1} & S\theta_{n+1}S\alpha_{n+1} & a_{n+1}C\theta_{n+1} \\ S\theta_{n+1} & C\theta_{n+1}C\alpha_{n+1} & -C\theta_{n+1}S\alpha_{n+1} & a_{n+1}S\theta_{n+1} \\ 0 & S\alpha_{n+1} & C\alpha_{n+1} & d_{n+1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DENAVIT-HARTENBERG REPRESENTATION OF FORWARD KINEMATIC EQUATIONS OF ROBOT

$$A_{n+1} = \begin{bmatrix} C\theta_{n+1} & -S\theta_{n+1}C\alpha_{n+1} & S\theta_{n+1}S\alpha_{n+1} & a_{n+1}C\theta_{n+1} \\ S\theta_{n+1} & C\theta_{n+1}C\alpha_{n+1} & -C\theta_{n+1}S\alpha_{n+1} & a_{n+1}S\theta_{n+1} \\ 0 & S\alpha_{n+1} & C\alpha_{n+1} & d_{n+1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- For example, the transformation between joints 2 and 3 of a generic robot is:

$$A_3 = \begin{bmatrix} C\theta & & & \\ S\theta & & & \\ 0 & & & \\ 0 & & & \end{bmatrix}$$

Table 2.1 D-H parameters table.

#	θ	d	a	α
0-1				
1-2				
2-3				
3-4				
4-5				
5-6				

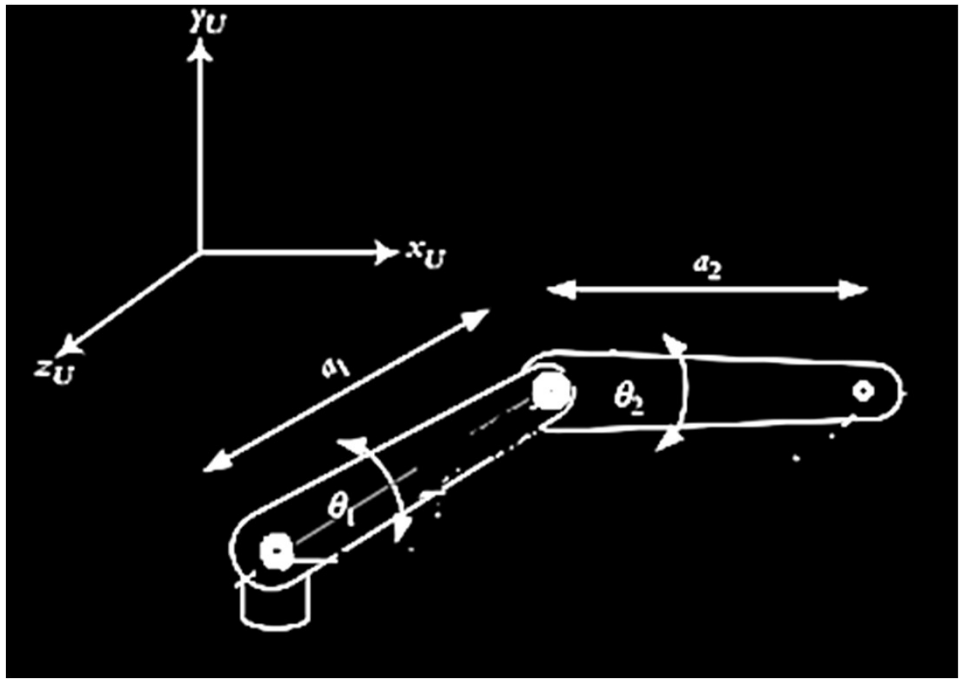
- It is expected to have one … sformation an A_{n+1}, the total transformati… bot is:

$$^{R}T$$

- To facilitate the calculatio… s table where joint and link parameters from the schematic drawing of the robot are determined. Subsequently, these values are substituted into each A matrix.

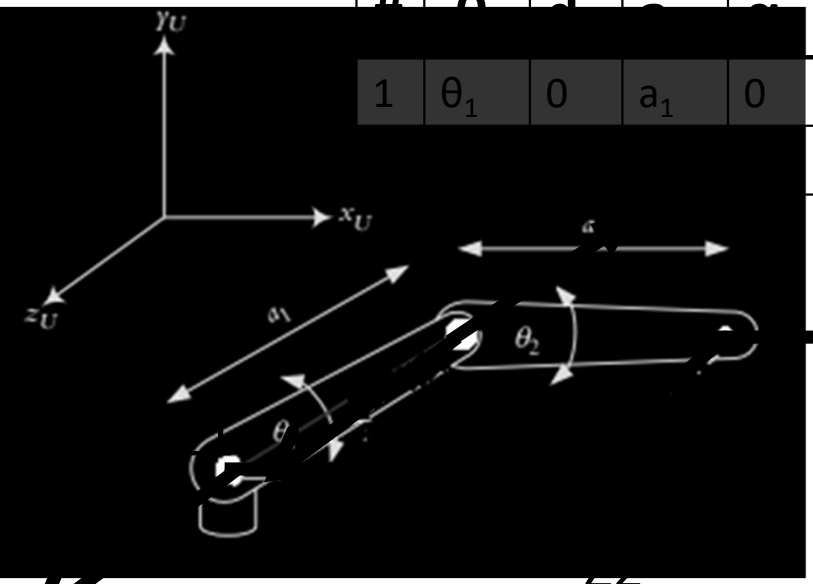
Numerical Problems: D-H procedure

- For the simple 2-axis, planar robot of shown below, assign the necessary coordinate systems based on the D-H representation, fill out the parameters table, and derive the forward kinematic equations for the robot.



Numerical Problems: D-H procedure

#	θ	d	a	α
1	θ_1	0	a_1	0



x2

z2

z0

$$A_{n+1} = \begin{bmatrix} C\theta_{n+1} & -S\theta_{n+1}C\alpha_{n+1} & S\theta_{n+1}S\alpha_{n+1} & a_{n+1}C\theta_{n+1} \\ S\theta_{n+1} & C\theta_{n+1}C\alpha_{n+1} & -C\theta_{n+1}S\alpha_{n+1} & a_{n+1}S\theta_{n+1} \\ 0 & S\alpha_{n+1} & C\alpha_{n+1} & d_{n+1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

I) Rotate about the z_n-axis an angle of θ_{n+1}. This will make x_n parallel to x_{n+1}. (Coplanar)

((II) Translate along z_n-axis a distance of d_{n+1} to make x_n and x_{n+1} colinear.

(III) Translate along the x_n-axis a distance of a_{n+1} to bring the origins of x_n and x_{n+1} together.

IV) Rotate z_n-axis about x_{n+1} axis an angle of α_{n+1} to align z_n-axis with z_{n+1}-axis.

$$A_1 = \begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

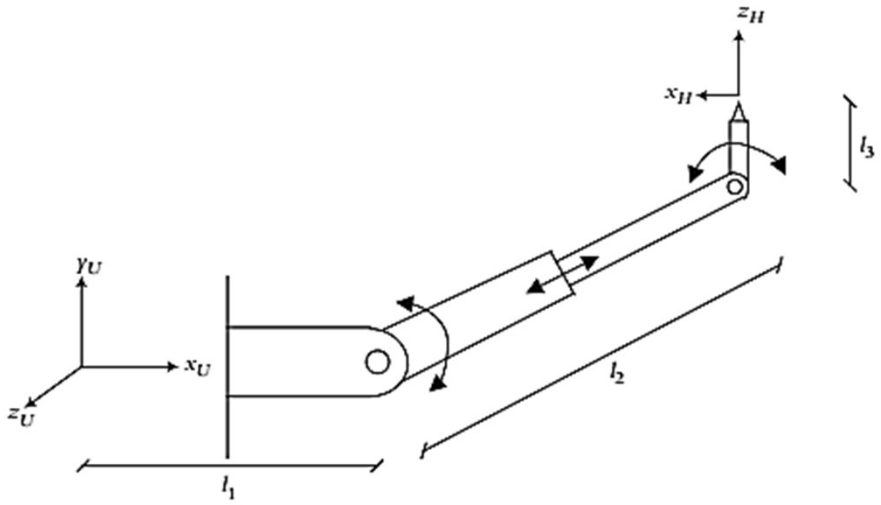
$$A_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = A_1XA_2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & a_2C_{12+}a_1C_1 \\ S_{12} & C_{12} & 0 & a_2S_{12} + a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

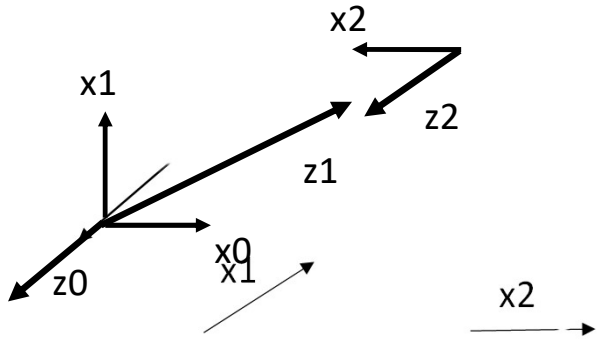
Numerical Problems: D-H procedure

A special 3-DOF spraying robot has been designed as shown:

1. Assign the coordinate frames based on the *D-H* representation.
2. Fill out the parameters table.
3. Write all the *A* matrices. And Write the ${}^{U}T_{H}$ matrix in terms of the *A* matrices.



Numerical Problems: D-H procedure



#	θ	d	a	α
1	$90+\theta_1$	0	0	90
2	θ	l_2	0	-90
3	θ_3	0	0	90

$$A_{n+1} = \begin{bmatrix} C\theta_{n+1} & -S\theta_{n+1}C\alpha_{n+1} & S\theta_{n+1}S\alpha_{n+1} & a_{n+1}C\theta_{n+1} \\ S\theta_{n+1} & C\theta_{n+1}C\alpha_{n+1} & -C\theta_{n+1}S\alpha_{n+1} & a_{n+1}S\theta_{n+1} \\ 0 & S\alpha_{n+1} & C\alpha_{n+1} & d_{n+1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} -S_1 & 0 & C_1 & 0 \\ C_1 & 0 & S_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} -S_3 & 0 & C_3 & 0 \\ C_3 & 0 & S_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_0 = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = A_0.A_1.A_2.A_3.A_4$$

The Inverse Kinematic Solution of Robots

- The forward kinematic equation for the robot is given below it is desired to place the robot at a position and consequently, an orientation - given as n, o, a, p vectors:

•

$$\begin{bmatrix} C_{12} & -S_{12} & 0 & a_2C_{12}+a_1C_1 \\ S_{12} & C_{12} & 0 & a_2S_{12}+a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Inverse Kinematic Solution of Robots

Solution:

$$\begin{bmatrix} C_{12} & -S_{12} & 0 & a_2C_{12}+a_1C_1 \\ S_{12} & C_{12} & 0 & a_2S_{12}+a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_{12} = n_y \text{ and } C_{12} = n_x \rightarrow \theta_{12} = ATAN2(n_y, n_x)$$

$$a_2C_{12} + a_1C_1 = p_x \text{ or } a_2n_x + a_1C_1 = p_x \rightarrow C_1 = \frac{p_x - a_2n_x}{a_1}$$

$$a_2S_{12} + a_1S_1 = p_y \text{ or } a_2n_y + a_1S_1 = p_y \rightarrow S_1 = \frac{p_y - a_2n_y}{a_1}$$

$$\theta_1 = ATAN2(S_1, C_1) = ATAN2\left(\frac{p_y - a_2n_y}{a_1}, \frac{p_x - a_2n_x}{a_1}\right)$$

Since θ_1 and θ_{12} are known, θ_2 can also be calculated.

POSITION REPRESENTATION

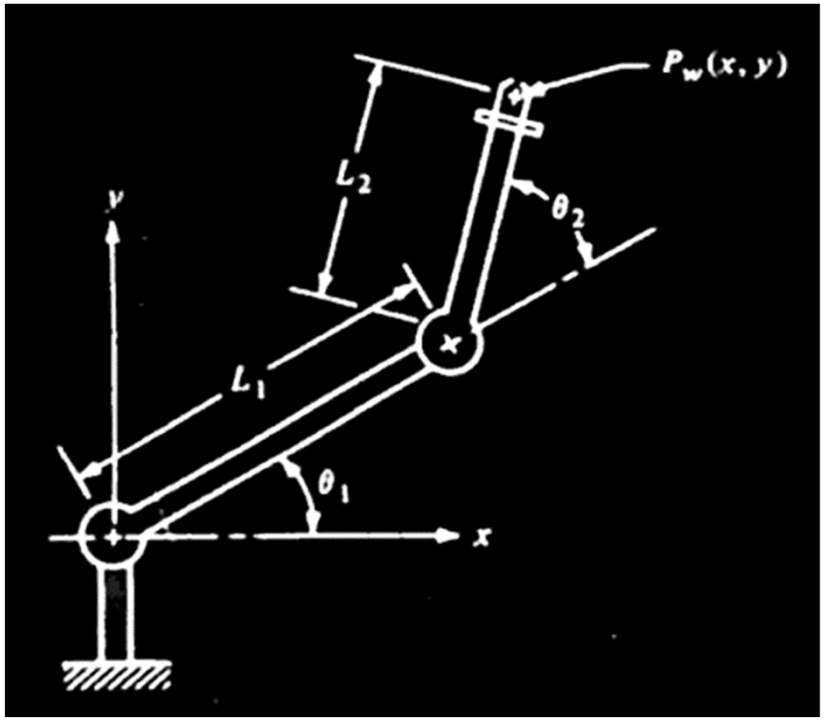
- The end arm position of a robot can be represented in two ways:
  - **Representation in joint space :** $P_j = (\theta_1 , \theta_2)$
  - **Representation in world space :** $P_w = (x , y)$ or $P_w = (x , y , z)$

- Joint space ⟶ World space

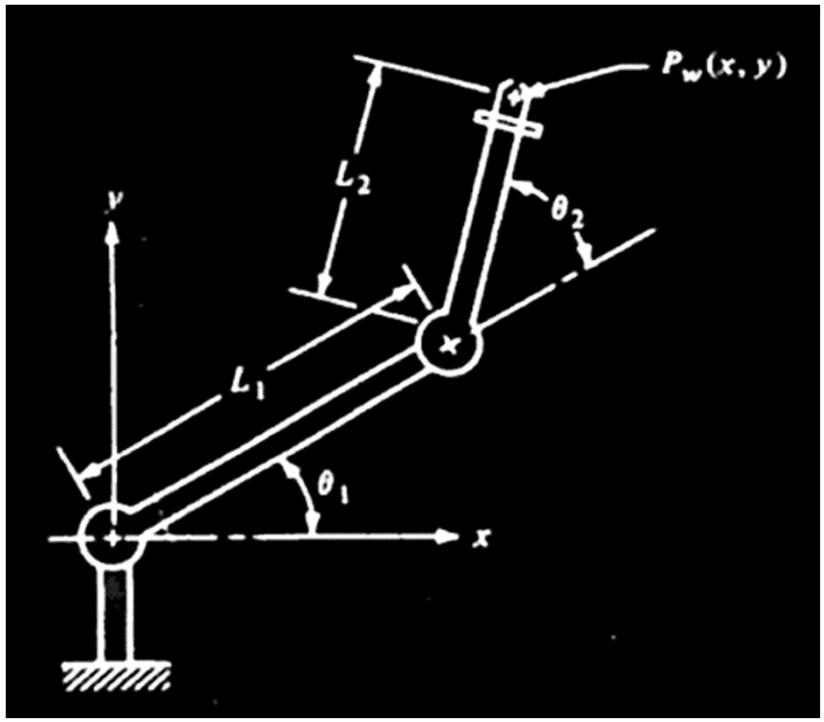
  Forward transformation or Forward kinematics.

- World space ⟶ Joint space

  Reverse transformation or inverse kinematics.



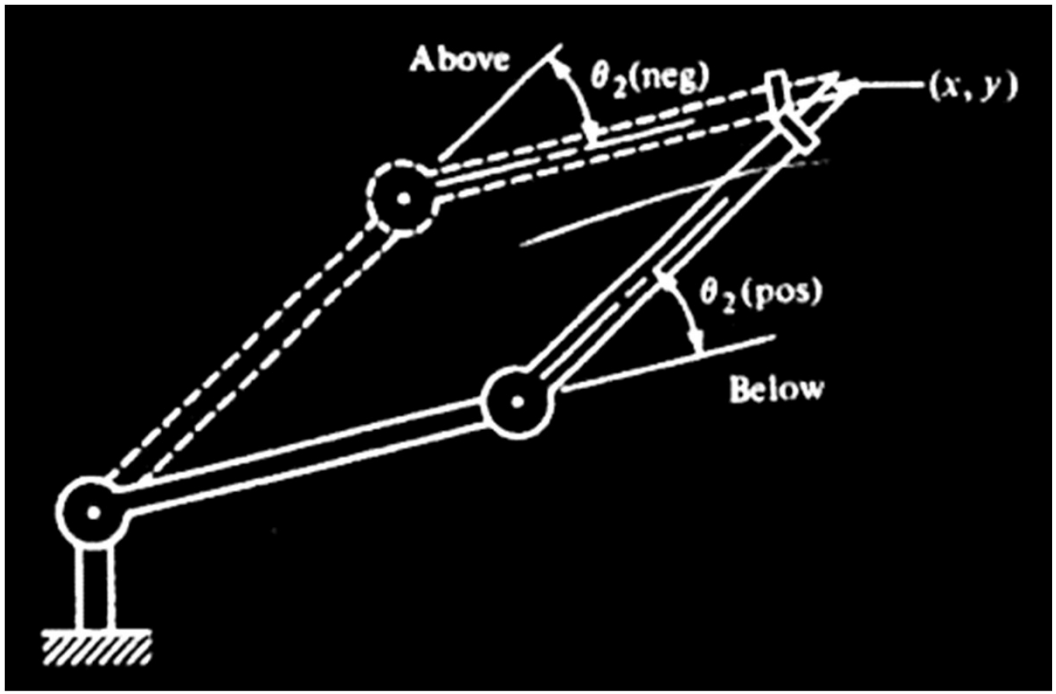
FORWARD TRANSFORMATION OF 2 DOF ARM



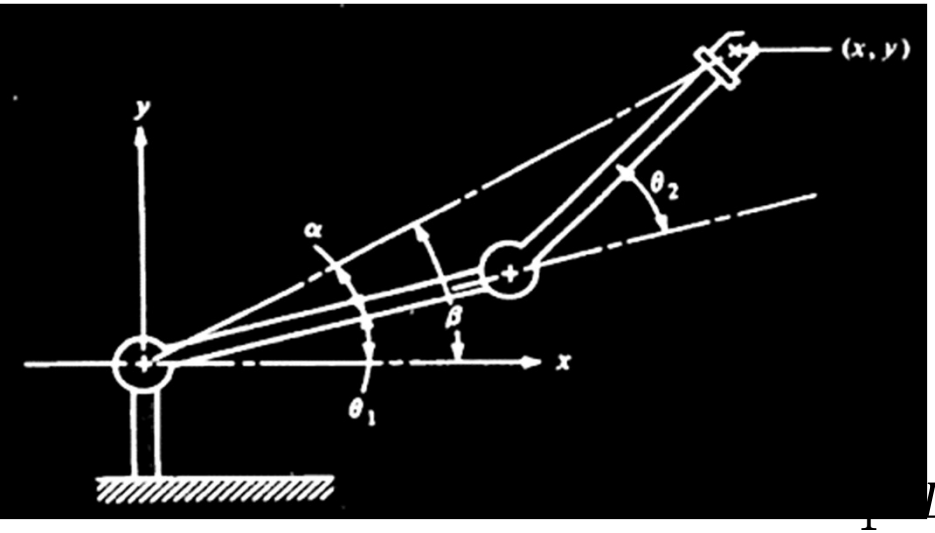
$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$= L_1 c_1 + L_2 c_{12}$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$
$$= L_1 s_1 + L_2 s_{12}$$

REVERSE TRANSFORMATION OF 2 DOF ARM

- There are two ways to reach the point (x , y) as shown in the figure:



REVERSE TRANSFORMATION OF 2 DOF ARM



$$x = L_1 c_1 + L_2 c_{12}$$
$$y = L_1 s_1 + L_2 s_{12}$$

$$= (L_1 c_1 + L_2 c_{12})^2 = L_1^2 c_1^2 + L_2^2 c_{12}^2 + 2L_1 L_2 c_1 c_{12}$$
$$= (L_1 s_1 + L_2 s_{12})^2 = L_1^2 s_1^2 + L_2^2 s_{12}^2 + 2L_1 L_2 s_1 s_{12}$$

$$+ L_1^2 s_1^2 + L_2^2 c_{12}^2 + L_2^2 s_{12}^2 + 2L_1 L_2 c_1 c_{12} + 2L_1 L_2 s_1 s_{12}$$

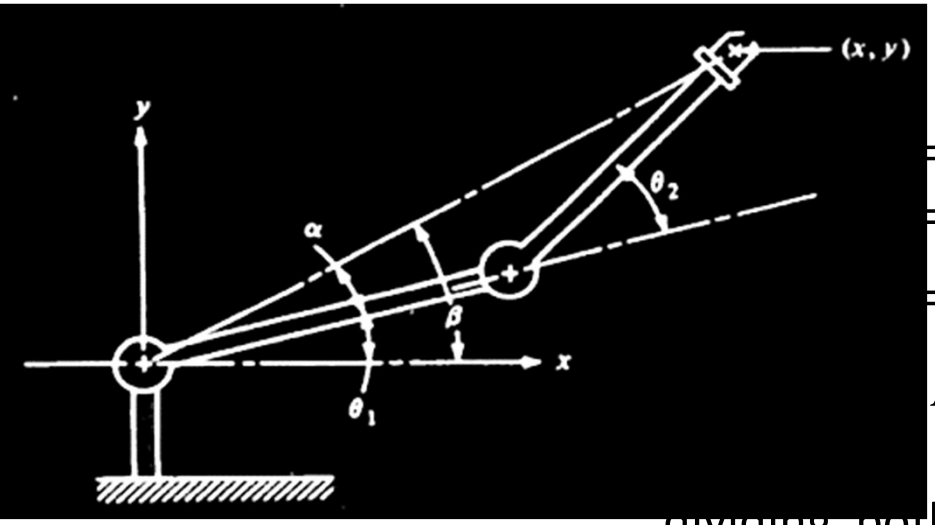
$$L_2^2 + 2L_1 L_2 (c_1 c_{12} + s_1 s_{12})$$

$$= L_1^2 + L_2^2 + 2L_1 L_2 c_2$$

$$c_2 = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1 L_2}$$

$$\theta_2 = \cos^{-1}\left(\frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1 L_2}\right)$$

REVERSE TRANSFORMATION OF 2 DOF ARM



$$x = L_1 c_1 + L_2 c_{12} \qquad k_1 = L_1 + L_2 c_2$$
$$y = L_1 s_1 + L_2 s_{12} \qquad k_2 = L_2 s_2$$

$$= L_1 c_1 + L_2(c_1 c_2 - s_1 s_2) = (L_1 + L_2 c_2) c_1 - L_2 s_2 s_1$$
$$= k_1 c_1 - k_2 s_1$$
$$= L_1 s_1 + L_2(s_1 c_2 + c_1 s_2) = k_1 s_1 + k_2 c_1$$

Let $\tan\alpha = \frac{K_2}{k_1}$ $\qquad \frac{y}{x} = \frac{k_1 s_1 + k_2 c_1}{k_1 c_1 - k_2 s_1}$

dividing both the numerator and denominator by $k_1 c_1$, results in

$$\frac{y}{x} = \frac{\tan\theta_1 + \tan\alpha}{1 - \tan\theta_1 \tan\alpha} = \tan(\theta_1 + \alpha)$$

or $\theta_1 + \alpha = \tan^{-1}\left(\frac{y}{x}\right) \quad \therefore \theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{k_2}{k_1}\right)$

$$= \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{L_2 s_2}{L_1 + L_2 c_2}\right)$$

Numerical Problem

Perform inverse kinematics and find all possible sets of θ_1 and θ_2 in a planar RR type robot with two degrees of freedom, to locate the end effector at (3,4 meters). Take the link lengths l_1 = 4 meters and l_2 = 2 meters.

Solution

$$Given: x = 3m;\ y = 4m; l_1 = 4m\ and\ l_2 = 2m$$

$$\theta_2 = \cos^{-1}\left(\frac{x^2+y^2-L_1^2-L_2^2}{2L_1L_2}\right) = \cos^{-1}\left(\frac{3^2+4^2-4^2-2^2}{2\times4\times2}\right) = \cos^{-1}(0.3125) = 71.79^o\ or\ 288.21^o$$

$$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{L_2s_2}{L_1+L_2c_2}\right) = \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{2\times\sin 71.79^o\ or\ 288.21^O}{4+2\times\cos 71.79^o\ or\ 288.21^o}\right)$$

$$= 53.13^o - 22.33^o\ or\ -22.33^o = 30.79^o\ or\ 75.46^o$$

3 DOF ARM IN TWO DIMENSIONS

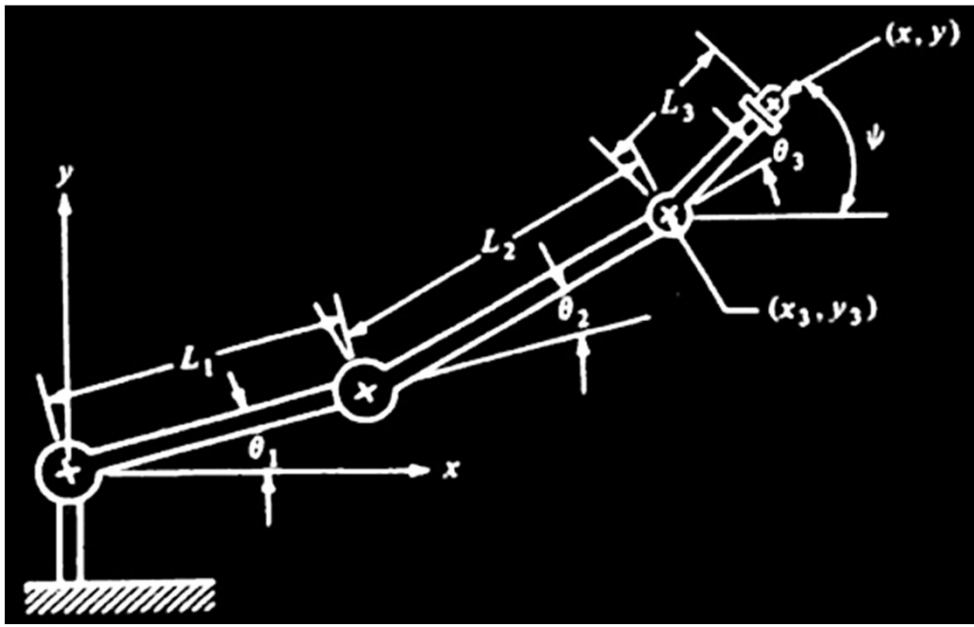
- A third degree of freedom can be achieved by incorporating one more link into a two degree of freedom arm. This joint will have a rotational movement as shown in the Figure.
- The world space coordinates for the wrist end are as follows:

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

$$\psi = \theta_1 + \theta_2 + \theta_3$$

- The joint space coordinates can be calculated from:
- $x_3 = x - L_3 \cos\psi$
- $y_3 = y - L_3 \sin\psi$



4 DOF ARM IN THREE DIMENSIONS

- The world space coordinates for the wrist end are as follows:

$$x = \cos\theta(L\cos\phi + L_4\cos\psi)$$
$$y = \sin\theta(L\cos\phi + L_4\cos\psi)$$
$$z = L_1 + L\sin\phi + L_4\sin\psi$$

- The joint space coordinates can be calculated from:

$$x_4 = x - \cos\theta(L_4\cos\psi)$$
$$y_4 = y - \sin\theta(L_4\cos\psi)$$
$$z_4 = z - L_4\sin\psi$$

- The values of L, ϕ, θ can be calculated:

$$L = [x_4^2 + y_4^2 + (z_4 - L_1)^2]^{-1}$$

$$\sin\phi = \frac{z_4 - L_1}{L}$$

$$\cos\theta = \frac{y_4}{L}$$

