# Partial ordering (UNIT-III)

2Y CSE-SI
DMS

3.1

## Partial ordering:

A binary relation 'R' in a set 'P' is called "partial ordering relation" or "partial ordering" in 'P' if and only if R is reflexive, antisymmetric and transitive.

→ we denote a partial ordering by the symbol $\leq$.

If $\leq$ is a partial ordering on 'A', then the order pair $(P, \leq)$ is called a "partial order set" or "poset"

## Totally ordered:

let $(P, \leq)$ be a poset. If for every $x, y \in P$ we have either $x \leq y \vee y \leq x$, then $\leq$ is called a simple ordering or linear ordering on P. and $(P, \leq)$ is called totally ordered or simply ordered set or chain.

Show that "greater than or equal" relation $(\geq)$ is a partial ordering on the set of integers.

Solution:

since $a \geq a$ for every integer $a$, $\geq$ is reflexive.

if $a \geq b$ and $b \geq a$, then $a = b$, hence $\geq$ is antisymmetric.

if $a \geq b$ and $b \geq c$ imply that $a \geq c$ hence $\geq$ is transitive.

it follows that $\geq$ is a partial ordering on the set of integers and "$(Z, \geq)$" is a poset.

→ In a poset $(P, \leq)$ an element $y \in P$ is said " to "cover" an element $x \in P$ if $x < y$ and if there does not exist an element $z \in P$ such that $x \leq z$ and $z \leq y$ that is

$$y \text{ covers } x \iff (x < y \wedge (x \leq z \leq y \Rightarrow x = z \vee z = y))$$

**Hasse Diagram:-** (German mathematician) → Helmut Hasse.

A partial ordering $\leq$ on set $P$ can be represented by a diagram known as hasse diagram of $(P, \leq)$ (or) poset diagram of $(P, \leq)$.

The procedure for drawing Hasse diagram for a poset 'P' as follows:

① Each element is represented by "small circle."

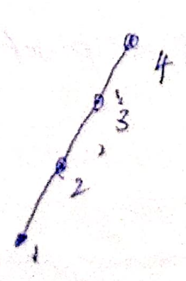
② The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$.

③ A line is drawn between $x$ and $y$ if $y$ covers $x$ and if $x < y$ but $y$ does not cover $x$, then $x$ and $y$ are not connected directly by a single line.

It is possible to obtain the set of ordered pair in $\leq$ from such diagram.

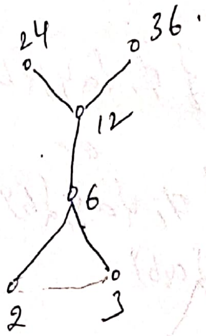
**Ex:** Let $P = \{1, 2, 3, 4\}$ and $\leq$ be the relation "less than or equal to" then the Hasse diagram is.

**Sol:-** $1 \leq 2 \leq 3 \leq 4$.



Ex:- let $x = \{2, 3, 6, 12, 24, 36\}$ and the relation $\preceq$ be such that $x \preceq y$ if $x$ devides $y$.

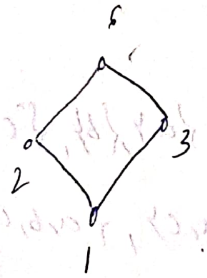
Sol:- Hasse diagram.




Ex: let $D_6 = \{1, 2, 3, 6\}$. Draw Hasse Diagram of $(D_6, |)$.

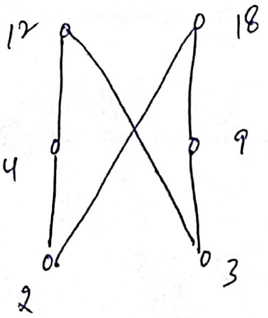
Hasse Diagram is.

Sol:-




Ex:- let $(\{2, 3, 4, 9, 12, 18\}, |)$ be a poset. Draw its Hasse Diagram.

Sol:-




Example: let A be any finite set and $P(A)$ be the power set of $(A)$. $\subseteq$ be the inclusion relation on the elements of $P(A)$. Draw Hasse diagrams of $(P(A), \subseteq)$ for

(i) $A = \{a\}$

(ii) $A = \{a, b\}$

(iii) $A = \{a, b, c\}$

(iv) $A = \{a, b, c, d\}$.

$P(A)$

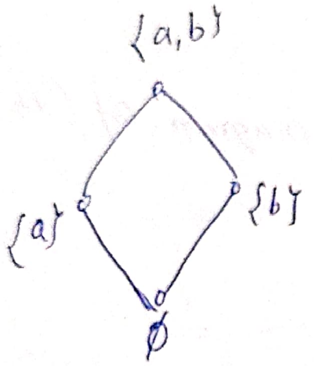
(i) Given $A = \{a\} \Rightarrow P(A) = \{\phi, \{a\}\}$

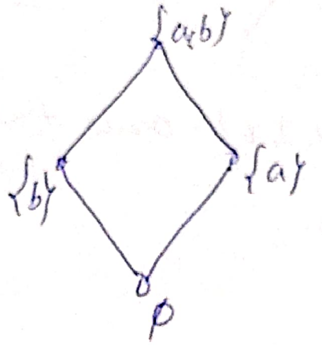
Diagram is


(ii) Given $A = \{a, b\} \Rightarrow P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$.

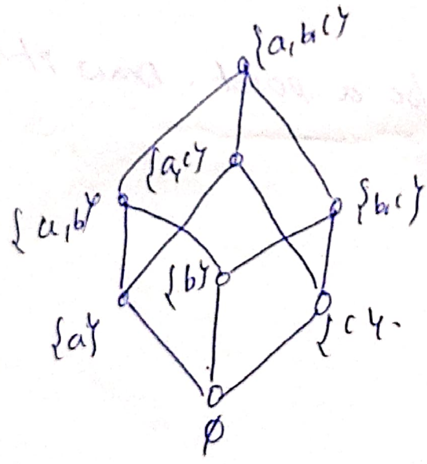
or

$P(A) = \{\phi, \{a\}, \{b\}, \{b, a\}\}$.



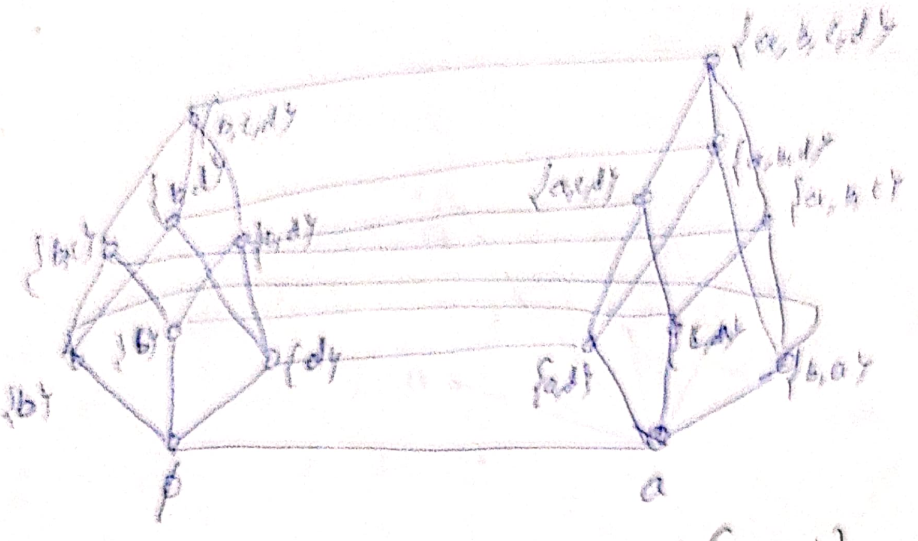




(iii) Given $A = \{a, b, c\} \Rightarrow P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.



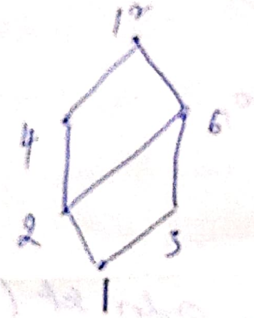

(iv) Given $A = \{a, b, c, d\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c, d\}\}$.



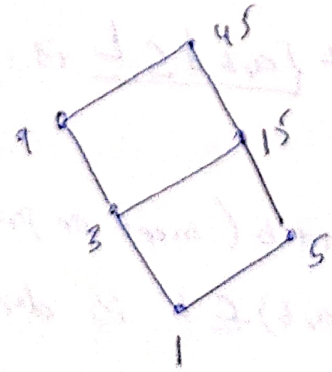

Ex:- Draw Hasse diagram of poset $(D_{12}, |)$.

Sol:- $D_{12} = \{1, 2, 3, 4, 6, 12\}$.




Ex:- Draw the Hasse diagram of poset $(D_{45}, |)$.

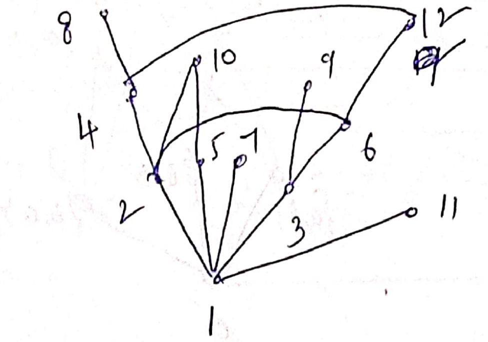
Sol:- $D_{45} = \{1, 3, 5, 9, 15, 45\}$.



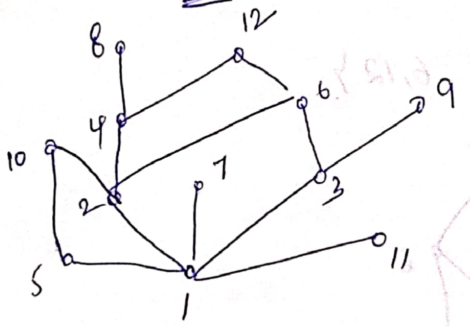

Ex:- Let $I_{12}$ is the set of all positive integers which are less than or equal to 12. Draw the Hasse diagram of $(I_{12}, |)$.

Sol:- $I_{12} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Hasse Diagram




(or)




$\{a,b\} \subseteq L$

$a*b$

$a \wedge b$

$a \oplus b$

$a \vee b$

**Lattices:-**

A Lattice is a partially order set $(L, \leq)$ in which every pair of elements $a, b \in L$ has greatest lower bound (GLB) and least upper bound (LUB).

→ The GLB of a subset $\{a,b\} \subseteq L$ is denoted by $a*b$ (or) $a \wedge b$.

i.e $GLB\{a,b\} = a*b$ (meet or product of a & b).

→ The LUB of a subset $\{a,b\} \subseteq L$ is denoted by $a \oplus b$ or $a \vee b$.

i.e $LUB\{a,b\} = a \oplus b$ (join or sum of a & b)

→ The lattice is an algebraic system with binary operations $\wedge$ and $\vee$. it is denoted by.

$[L, \wedge, \vee]$ or $[L, *, \oplus]$.

⇒. Let $(P, \leq)$ be a poset and $a, b \in P$.

(i) upper bound:-

An element $c \in P$ is an upper bound of $a$ & $b$ in $P$

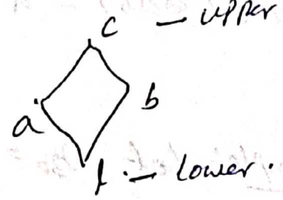
if $a \leq c$ and $b \leq c$.

(ii) Lower bound:-

An element $l \in P$ is a lower bound of $a$ & $b$ in $P$ if

$l \leq a$ and $l \leq b$.

Ex:-




(iii) least upper bound (LUB):-

An element $u \in P$ is a least upper bound of $a$ & $b$ in $P$.

if (i) $u$ is an upper bound of $a$ & $b$.

(ii) $u$ is the least among all upper bounds of $a$ & $b$.

i.e if there exist an element $u_1 \in P$ such that

$a \leq u_1$, $b \leq u_1$ then $u \leq u_1$.

(iv) Greatest lower bound (GLB)

An element $l \in P$ is called a greatest lower bound

of $a$ & $b$ in $P$. if

(i) $l$ is lower bound of $a$ & $b$.

(ii) $l$ is the greatest among all lower bounds of

$a$ & $b$. i.e if there exist an element $l_1 \in P$

such that $l_1 \leq a$, $l_1 \leq b$ then $l_1 \leq l$.

Here we write:-

$a \wedge b = a * b = l$ = glb of $a$ and $b$.

$a \vee b = a \oplus b = u$ = lub of $a$ and $b$.

$(L, \leq)$.

LUB
GLB

## Properties of Lattice :

Let $(L, \leq)$ be lattice, then for $a, b \in L$.

(i) $a \vee a = a$ and $a \wedge a = a$ (idempotent law)

(ii) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$. (Commutative law)

(iii) $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$. – (associative law)

(iv) $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ – (absorption law).

**Example:** Let $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and let the relation 'D' be divisor on $D_{30}$ (i.e $xRy$ if $x$ divides $y$).

**Find:-**

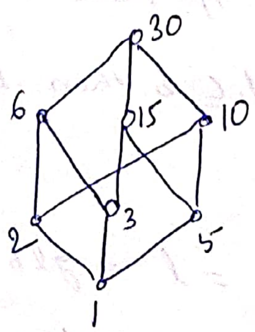
1. all lower bounds of 10 and 15.
2. all glb of 10 and 15.
3. all upper bounds of 10 and 15.
4. The lub of 10 & 15.
5. Draw the Hasse Diagram for $D_{30}$ with D.

**Sol:-** In $D_{30}$, the relation D is divisor.

i.e $xRy = x$ divides $y$.

Since '1' divides every number in $D_{30}$ then '1' is the least element.

**Hasse Diagram:-**




Every number in $D_{30}$ divides 30.

$\Rightarrow$ 30 is the greatest element.

$\therefore (D_{30}, |)$ is a Lattice.

① All lower bounds of 10 and 15 are.

1 and 5.

② All glb of 10 and 15 is. 5

③ upper bounds of 10 and 15 is 30

④ The Lub of 10 and 15 is 30

Example:- Let $L = \{1, 3, 5, 7, 15, 21, 35, 105\}$ and let $\leq$ be the relation divides on L.

find:-

① upper bound of 3 & 7

② lub of 3 & 7.

③ lower bound of 15 & 21

④ glb of 15 & 21.

⑤ upper bonds and lower bonds of 15 & 35.

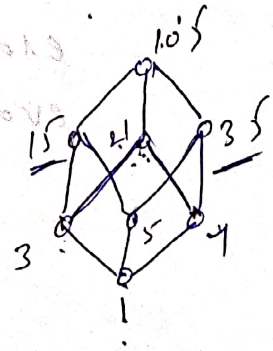
⑥ Lub & glb of 15 & 35.

Sol:- ~~Since~~ since 1 divides every element in L. then '1' is the least element.

Every Element in L divides 105.

$\Rightarrow$ 105 is the greatest element.

$(L, /)$ is a Lattice.



① upper bounds of 3 and 7 ⇒ 21, 105

② lub of 3 & 7 ⇒ 21.

③ lower bounds of 15 & 21 ⇒ 1, 3.

④ glb of 15 & 21 ⇒ 3

⑤ upper bounds of 15 & 35 ⇒ 105

lower bounds of 15 & 35 ⇒ 1, 5

⑥ lub of 15 & 35 ⇒ 105

glb of 15 & 35 ⇒ 5.

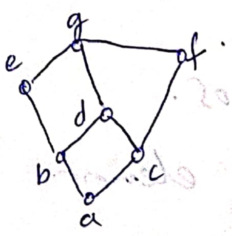
## Sub Lattice:

Let $(L, \vee, \wedge)$ be a lattice. A subset M of L is a sub lattice, if

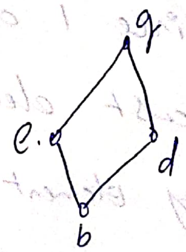
$x, y \in M \Rightarrow x \vee y \in M$ and $x \wedge y \in M$.

Example:

Lattice




Sub lattice

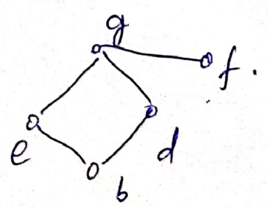



because it has lower bound and upper bound.

$e \wedge d = b$

$e \vee d = g$.

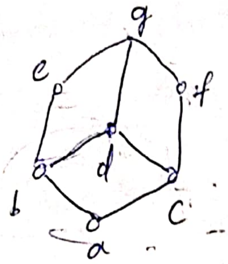
not a sub lattice



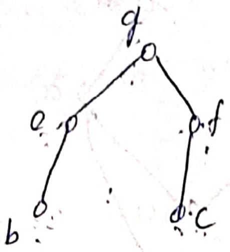

since $d \wedge f = c$ is not here.

Ex: Consider the following lattices L. and the partially order sub set of L shown. find out subsets S of L are sub lattices or not.

Sol:-

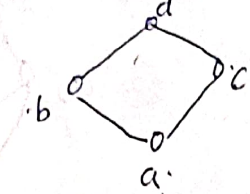



Lattice




$b \wedge e \notin S$.
$b \vee c \notin S$

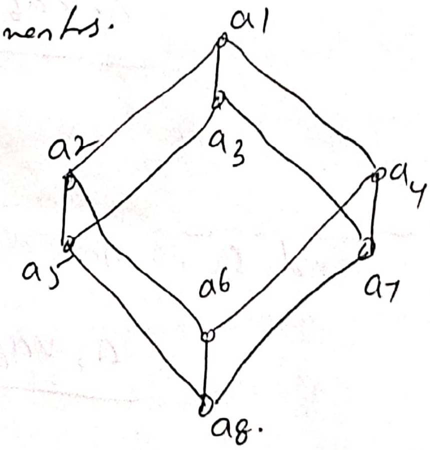
not a sub lattice




$b \wedge c = a \in S$
$b \vee a = d \in S$.

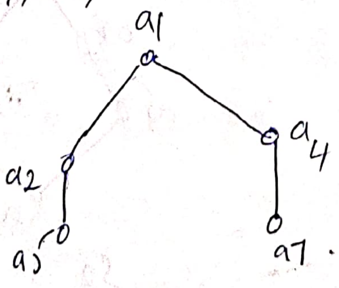
Sub Lattice

Example: Consider the lattice 'L' is shown figure, find four sub lattices with five elements.

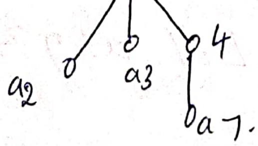



Sol: The sub lattices of L are.

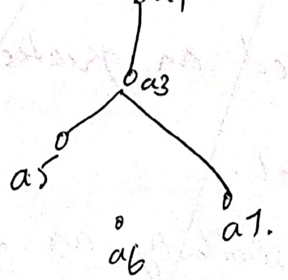
$S_1 = \{a_1, a_2, a_4, a_5, a_7.\}$.



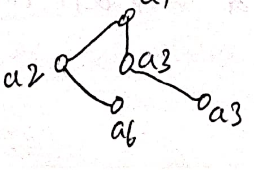

$S_2 = \{a_1, a_2, a_3, a_4, a_7\}$.




$S_3 = \{a_1, a_3, a_5, a_6, a_7\}$

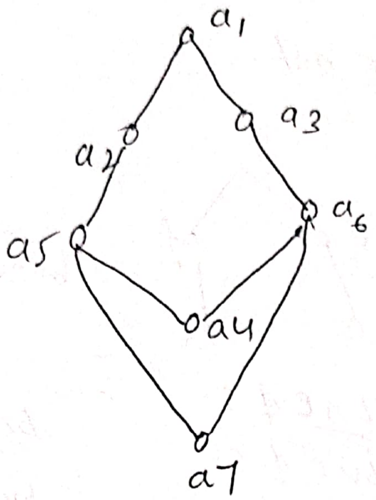



$S_4 = \{a_1, a_2, a_3, a_6, a_7\}$.



Example:

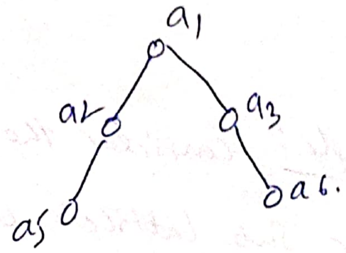
Consider the lattice $L$ shown in fig. find posets of which are not sublattices of $L$.




solution: The Hasse Diagram of $S_1$ is not a sub lattice of $L$

because $a_5 \wedge a_6 \notin S_1$.

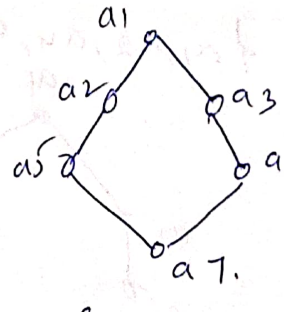
$a_5 \vee a_6 \notin S_1$.




$S_1$

and $S_2$ is not a sublattice of $L$ because

$a_5 \vee a_6 \notin S$.




$S_2$

## Special Lattices:

Let $(L, \leq)$ be a lattice, An element $g \in L$ is called an greatest element of $L$ if $a \leq g$ for all $a \in L$.

Similarly an ~~de~~ element $s \in L$ is called the least element of $L$ if $s \leq a$ for all $a \in L$.

Model

## Maximal and Minimal Elements

To understand the concept of maximal and minimal elements of partial ordered set, it is important to be familiar with the least and greatest members.

$a \wedge (b \vee a)$ (a is b) $a \wedge b$

### Least Member:-

Let $(P, \leq)$ denote a partially ordered set, if there exists an element $y \in P$ such that $y \leq x$ for $x \in P$, then 'y' is called least member in P.

### Greatest Member:-

Let $(P, \leq)$ denote a partially ordered set, if there exists an element $y \in P$ such that $x \leq y$ for all $x \in P$, then y is called the greatest member in P relative to $\leq$.

### Note:

① The least member and greatest member are unique and are usually denoted by '0' and '1' respectively.

② An element $x \in P$ is called a minimal member of P relative to a partial ordering $\leq$ if there is no $x \in P$ such that $x < y$.

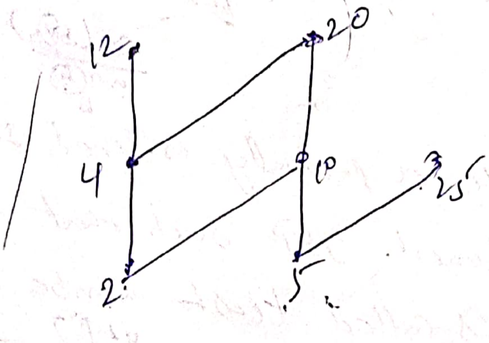
③ An element $y \in P$ is called a maximal member of relative to a partial ordering $\leq$ if there is no $x \in P$ such that $y < x$.

④ A maximal or minimal member need not be unique.

⑤ Maximal and minimal elements are easily calculated from the Hasse diagram. They are the top and bottom elements in the diagram.

Ex: which elements of the poset $(\{2,4,5,10,12,20,25\}, |)$ are maximal and which are minimal.

Sol: The Hasse diagram of $(\{2,4,5,10,12,20,25\}, |)$ is shown below.



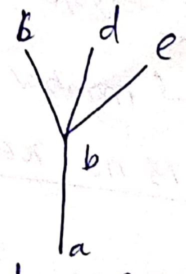

maximal elements are = 12, 20 and 25.

minimal elements are = 2 and 5.

Thus, a poset can have more than one maximal and more than one minimal element.

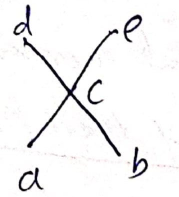
Ex: Determine whether a posets represented by each of the Hasse diagram in below have a greatest and a least element.

Sol:

(a) 

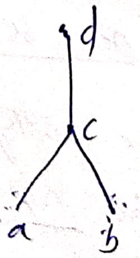

least element (LE) = a.

⇒ poset has no greatest element. (GE).

(b) 

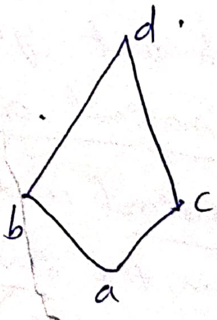

LE = 0

GE = 0

(c) 


LE = 0

GE = d.

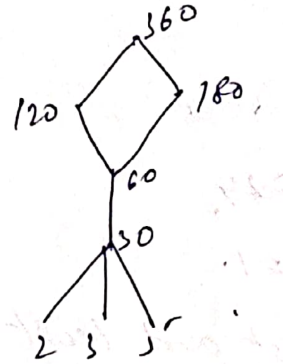
(d) 


LE = a

GE = d.

Ex:- Consider the poset $(\{2,3,5,30,60,120,180,360\}, |)$ having hasse diagram in below. Find GLB & LUB.

Sol:-




The set $\{120,180,360\}$ has glb $=\{60\}$.

but $\{2,3,5,30\}$ has no lower bounds and hence no glb.

The set $\{60,120,180\}$ has lub $=\{360\}$.

Example: let $D_{24}=\{1,2,3,4,6,8,12,24\}$ and the relation divides '|' be a partial ordering on $D_{24}$. Then the Hasse diagram of $(D_{24}, |)$ and also find the following.

(i) all lower bounds of 8,12

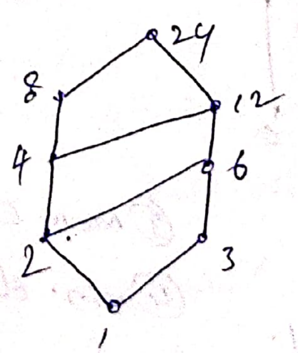
(ii) all upper bounds of 8,12

(iii) glb of 8,12

(iv) lub of 8,12.

(v) greatest and least elements of this poset if exist.

Sol:-



(i) The lower bounds of 8 and 12 are = {1,2,4}.

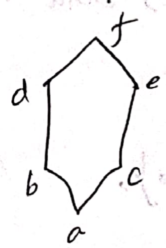
(ii) The upper bounds of 8 and 12 are = 24.

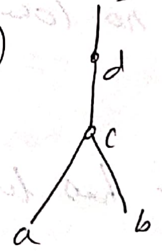
(iii) glb of 8,12 is 4.

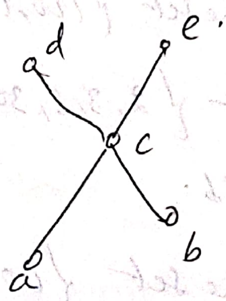
(iv) lub of 8,12 is 24.

(v) greatest element is = 24.
least element is = 1.

Example:- Determine the greatest and least elements of the following posets if they exists:

(i)



(ii)


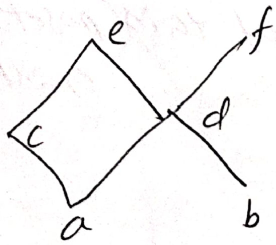

(iii)


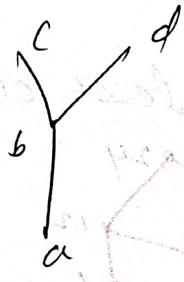

Sol: (i) greatest element is = f
least element is = a.

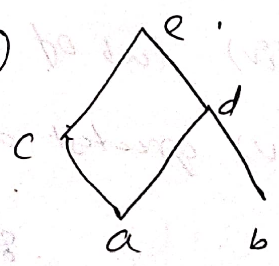
(ii) GE = e
LE = $\phi$.

(iii) GE = $\phi$
LE = $\phi$

Ex:- Determine the maximal and minimal elements of the posets.

(1)



(2)



(3)



Sol: (1) Max. Ele = e, f.
Min. Ele = a, b.

(2) Max. Elements = c, d.
Min. Elements = a.

(3) Max. Elements = e
(4) min. Elements = a, b.

**Ex:** Draw the poset diagram for each of the following posets and determine all maximal and minimal elements and greatest and least element if they exist.

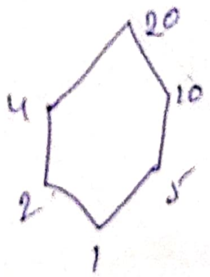
(i) $(D_{20}, /)$

(ii) $(D_{30}, /)$

(iii) $(A, /)$ where $A = \{2, 3, 4, 6, 8, 24, 48\}$.

(iv) $(A, /)$ where $A = \{2, 3, 4, 6, 12, 18, 24, 36\}$.

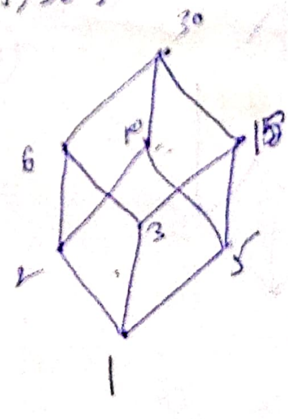
Sol: (i) $D_{20} = \{1, 2, 4, 5, 10, 20\}$.




min. Element $= 1 =$ LE

max. Element $= 20 =$ GE.

(ii) $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.



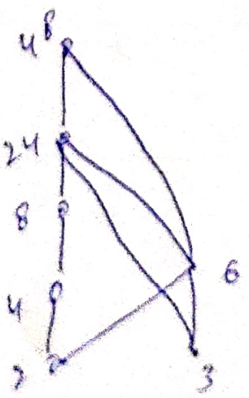

minimal element $= 1$

least Element $= 1$

maximal element $= 30$

Greatest Element $= 30$.

(iii) Given $A = \{2, 3, 4, 6, 8, 24, 48\}$.




minimal Elements $= 2, 3$.

least element = does not exist.

maximal element $= 48$

which is greatest element.

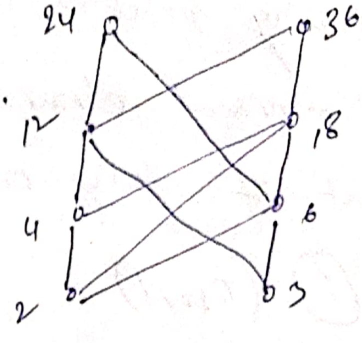
(iv) Given $A = \{2, 3, 4, 6, 12, 18, 24, 36\}$.

minimal elements = 2, 3.

least element = does not exist.

maximal elements = 24, 36.

Greatest element = does not exist.

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Ex: Determine whether the Hasse Diagrams represents a lattice?