UNIT III : Reccurence Relations

Def^n: A reccurence relation is an equation that recursively defines a sequence on a rule that gives next term in the sequence as a funcn of previous terms when one/more initial terms are given.

Recurrence Relations are two Types:

1) Linear Recurrence Relation/ Homogeneous RR
2) Non-Linear/Non- Homogeneous RR

To solve a Recurrence Relation, we have 3 approaches:

1) Iteration Method
2) Characteristic root Method
3) Generativeg Method

1) Find out the sequence generating by the recurrence relation given below

$$T_n = 2T_{n-1} \quad \text{with } T_1 = 4$$

Sol: $T_1 = 4$

$n = 1 \Rightarrow T_1 = 4$

$n = 2 \Rightarrow T_2 = 2 * T_1 = 2 \times 4$
$= 8$

$n=3, \quad T_3 = 2\cdot T_2 = 2\times 8 = 16$

$n=4, \quad T_4 = 2\cdot T_{4-1} = 2\times T_3 = 32$

$n=5, \quad T_5 = 2\cdot T_4 = 2\times 32 = 64$

$n=6, \quad T_6 = 2\cdot T_5 = 2\times 64 = 128$

$n=7, \quad T_7 = 2\cdot T_6 = 2\times 128 = 256$

Sequence of ~~pro~~ the generated terms: 4, 8, 16, 32, 64, 128 ....

Q) $T_n = T_{n-1} - 4$ with $T_1 = 3$

Sol: $n=1, \quad T_1 = 3$

$n=2, \quad T_2 = 3\cdot T_{2-1} - 4 = 9 - 4$

$= 5$

$n=3, \quad T_3 = 3\cdot T_{3-1} - 4 = 3\times 5 - 4$

$T_3 = 11$

$n=4 \Rightarrow T_4 = 3\cdot T_{4-1} - 4 = 3\times 11 - 4$

$= 29$

$n=5 \Rightarrow T_5 = 3\times T_{5-1} - 4 = 3\times 29 - 4$

$= 83$

$n=6 \Rightarrow T_6 = 3 \times T_{6-1} - 4 = 3 \times 83 - 4$

$= 245$

Sequence: 3, 5, 11, 29, 83, 245 . . .

Q) Find the reccurence relation for the following sequence

i) 2, 6, 18, 54, 162 . . .

Sol: Given sequence; 2, 6, 18, 54, 162

$T_1 = 2 \longrightarrow$ Initial Condition

$T_2 = 6 \Rightarrow$

$T_3 = 18$

$T_4 = 54$

$T_5 = 162$

⋮

$T_n = 3 * T_{n-1}$

ii) 20, 17, 14, 11, 8 . . . .

$T_1 = 20$ Initial Condition

$T_2 = 17 \Rightarrow T_{2-1} - 3 = T_1 - 3$

$T_3 = 14 \Rightarrow T_{3-1} - 3 = T_2 - 3$

$T_4 = 11 \Rightarrow T_{4-1} - 3 = T_3 - 3$

⋮

$T_n = T_{n-1} - 3$

iii) 1, 3, 6, 10, 15, 21 . . . .

$T_1 = 1$ Initial Condition

$T_2 = 3 \Rightarrow T_1 + 2$

$T_3 = 6 \Rightarrow T_2 + 3$

$T_4 = 10 \Rightarrow T_3 + 4$

$T_5 = 15 \Rightarrow T_4 + 5$

$\vdots$

$T_n = T_{n-1} + n$

Q) Find the first 5 terms of sequence defined by each of the following RR & initial cond^n.

i) $a_n = a_{n-1}^2$, $a_1 = 2$

$a_1 = 2$

$a_2 = a_1^2 = 2 \times 2 = 4$

$a_3 = a_2^2 = 4 \times 4 = 16$

$a_4 = a_3^2 = 16 \times 16 = 256$

$a_5 = a_4^2 = 256 \times 256 = 65536$

ii) $a_n = n a_{n-1} + n^2 a_{n-2}$ with $a_0 = 1$, $a_1 = 1$

$a_0 = 1$

$a_1 = 1$

$a_2 = 2 \cdot a_1 + 2^2 \cdot a_0 = 2 \times 1 + 4 \times 1$

$= 6$

$a_3 = 3 \cdot a_2 + 3^2 \cdot a_1 = 3 \times 6 + 9 \times 1$

$= 27$

$a_4 = 4 \cdot a_3 + 4^2 \cdot a_2 = 4 \times 27 + 16 \times 6$

$= 204$

$a_5 = 5 \cdot a_4 + 5^2 \cdot a_3 = 5 \times 204 + 25 \times 27$

$= 1695$

iii) $a_n = a_{n-1} + a_{n-3}$ with $a_0 = 1$, $a_1 = 2$, $a_2 = 0$

$a_0 = 1$

$a_1 = 2$

$a_2 = 0$

$a_3 = a_2 + a_0 = 0 + 1 = 1$

$a_4 = a_3 + a_1 = 1 + 2 = 3$

$a_5 = a_4 + a_2 = 3 + 0 = 3$

Q) Determine whether the sequence a_n is a solution of recurrence relation $a_n = a_{n-1} + 2 \cdot a_{n-2} + 2n - 9$

i) if $a_n = -n + 2$

ii) if $a_n = 3(-1)^n + 2^n - n + 2$

Sol:

i) $a_n = -n+2$

$a_{n-1} = -(n-1) + 2 = -n + 3$

$a_{n-2} = -(n-2) + 2 = -n + 4$

R.H.S $\Rightarrow a_{n-1} + 2 \cdot a_{n-2} + 2n - 9$

$= (-n+3) + 2(-n+4) + 2n - 9$

$= -n + 3 + -2n + 8 + 2n - 9$

$= -n + 2$

$= a_n$

$\therefore a_n = a_{n-1} + 2 \cdot a_{n-2} + 2n - 9$

ii) $a_n = 3(-1)^n + 2^n - n + 2$

$a_{n-1} = 3(-1)^{(n-1)} + 2^{(n-1)} - (n-1) + 2$

$-3(-1)^n + 2^{n-1} - n + 3$

$a_{n-2} = 3(-1)^{n-2} + 2^{n-2} - (n-2) + 2$

$= 3(-1)^n + 2^{n-2} - n + 4$

R.H.S $\Rightarrow$ $a_{n-1} + 2 \cdot a_{n-2} + 2n - 9$

$$= -3(-1)^n + 2^{n-1} - n + 3 + 2\left(3(-1)^n + 2^{n-2} - n + 4\right)$$

$$= -3(-1)^n + 2^{n-1} - n + 3 + 6(-1)^n + 2^{n-1} - 2n + 8$$

$$= 3(-1)^n + 2^n - 3n + 2n + 11 - 9$$

$$= 3(-1)^n + 2^n - n + 2$$

$$= a_n$$

Q) Let $a_n = 2^n + 5(3^n)$ for $n = 0, 1, 2 \ldots$

i) find a_0, a_1, a_2, a_3 and a_4

ii) S.T $a_2 = 5a_1 - 6a_0$

$a_3 = 5a_2 - 6a_1$

iii) S.T $a_n = 5a_{n-1} - 6a_{n-2}$ $\forall$ integers n with $n \geq 2$

Sol: Given, $a_n = 2^n + 5(3^n)$

$n = 0 \Rightarrow a_0 = 2^0 + 5(3^0) = 1 + 5 = 6$

$n = 1 \Rightarrow a_1 = 2^1 + 5(3^1) = 2 + 5(3)$

$\boxed{a_1 = 17}$

$n = 2 \Rightarrow a_2 = 2^2 + 5(3^2) = 4 + 5(9)$

$\boxed{a_2 = 49}$

$n = 3 \Rightarrow a_3 = 2^3 + 5(3^3) = 8 + 5(27)$

$\boxed{a_3 = 143}$

$n = 4 \Rightarrow a_4 = 2^4 + 5(3^4) = 16 + 5(81)$

$\boxed{a_4 = 421}$

i)

$a_0 = 6$

$a_1 = 17$

$a_2 = 49$

$a_3 = 143$

$a_4 = 421$

ii) R.H.S $\Rightarrow 5a_1 - 6a_0$

$= 5(17) - 6(6)$

$= 85 - 36$

$= 49$

$= a_2$

$\therefore a_2 = 5a_1 - 6a_0$

R.H.S $\Rightarrow 5a_2 - 6a_1$

$= 5(49) - 6(17)$

$= 245 - 102$

$= 143$

$= a_3$

$\therefore a_3 = 5a_2 - 6a_1$

iii) $a_n = 5a_{n-1} - 6a_{n-2}$

$$a_n = 2^n + 5(3^n)$$

$$a_{n-2} = 2^{(n-2)} + 5(3^{n-2})$$

$$a_{n-1} = 2^{n-1} + 5(3^{n-1})$$

R.H.S $\Rightarrow$ $5a_{n-1} - 6a_{n-2}$

$$= 5\left[2^{n-1} + 5(3^{n-1})\right] - 6\left[2^{n-2} + 5(3^{n-2})\right]$$

$$= 5.2^{n-1} + 25(3^{n-1}) - 6.2^{n-2} - 30.3^{n-2}$$

$$= 5.2^{n-1} + 25(3^{n-1}) - \overset{3}{\cancel{6}}.\frac{2^{n-1}}{\cancel{2}} - \overset{10}{\cancel{30}}.\frac{3^{n-1}}{\cancel{3}}$$

$$= 2.2^{n-1} + 10.3^{n-1}$$

$$= 2^n + 15.3^{n-1}$$

$$= 2^n + 5(3^n)$$

$$= a_n$$

$$\therefore \quad a_n = 5a_{n-1} - 6a_{n-2}$$

Solving the Reccurrence Relation using Characteristic Root

Order of a Recurrence Relation: Difference between highest subscript and lowest subscript

Ex: $a_n + a_{n-1} = 0 \longrightarrow n-(n-1) = n-n+1 = 1$ Order

$a_n + a_{n-1} + a_{n-2} = 0$

$a_n + 2a_{n-1} = r^2$

Method of Characteristic Roots

Steps:

1) Write the characteristic Equation for the given rearrence relation

2) Find the roots and lets roots be $r_1, r_2, r_3 \ldots r_n$

3) If r_1 and r_2 are Real & Distinct then the solution is $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ where α_1 & α_2 are arbitrary constants.

4) If r_1 and r_2 are Real and Equal then the sol^n is

$$a_n = (\alpha_1 + \alpha_2 n) r^n$$

5) If r_1 and r_2 are Complex then the sol^n is

$$a_n = r^n(\alpha_1 \cos n\phi + \alpha_2 \cos n\phi)$$

Q) Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$ $a_0 = 1$, $a_1 = 0$

Sol: The given recurrence relation is

$$a_n = 5a_{n-1} - 6a_{n-2}$$

$$a_n - 5a_{n-1} + 6a_{n-2} = 0 \longrightarrow (1)$$

Order = 2

Characteristic Eqn. $r^2 - 5r + 6 = 0 \longrightarrow (2)$

$$(r-2)(r-3) = 0$$

$$r = 2, 3$$

$$r_1 = 2 \qquad r_2 = 3$$

The roots are real and distinct

Then the soln is: $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

$$\therefore a_n = \alpha_1 2^n + \alpha_2 3^n \longrightarrow (3)$$

Initial conditions are $a_0 = 1$, $a_1 = 0$

$n = 0 \Rightarrow$ $a_0 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot 3^0$

$$1 = \alpha_1 + \alpha_2 \longrightarrow (i)$$

$n = 1 \Rightarrow$ $\alpha_1 \cdot 2^1 + \alpha_2 \cdot 3^1 = a_1$

$$2\alpha_1 + 3\alpha_2 = 0 \longrightarrow (ii)$$

Solving (i) & (ii)

$\alpha_2 = 1 - \alpha_1$ $\Rightarrow$ $2\alpha_1 + 3(1-\alpha_1) = 0$

$$2\alpha_1 + 3 - 3\alpha_1 = 0$$

$$\boxed{\alpha_1 = 3}$$

$\alpha_2 = 1 - 3$

$\boxed{\alpha_2 = -2}$

Substitute α_1, α_2 in (3)

$\therefore$ The sol^n is $\boxed{a_n = 3(2^n) - 2(3^n)}$

$n = 0 \Rightarrow a_0 = 3(2^0) - 2(3^0) = 1$

$n = 1 \Rightarrow a_1 = 3(2^1) - 2(3^1) = 0$

2) Solve the following recurrence relation $a_n - 8a_{n-1} + 16a_{n-2} = 0$ for $n \geq 2$ with $a_0 = 16$, $a_1 = 80$

Sol: Given recurrence relation:

$$a_n - 8a_{n-1} + 16a_{n-2} = 0$$

Order = 2

Ch. Eq: $r^2 - 8r + 16 = 0$

$$(r-4)(r-4) = 0$$

$$r = 4, 4$$

The roots are real and equal.

$$a_n = (\alpha_1 + \alpha_2 n) 4^n$$

$n = 0 \Rightarrow (\alpha_1 + \alpha_2 \cdot 0) 4^0 = a_0$

$\alpha_1 = 16$

$n = 1 \Rightarrow (\alpha_1 + \alpha_2) 4^1 = a_1$

$(16 + \alpha_2) \cancel{4} = \overset{20}{\cancel{80}}$

$\alpha_2 = 4$

$$\therefore a_n = (16 + 4n) 4^n$$

3.) Solve the following: $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$ with $a_0 = 3$, $a_1 = 6$ & $a_2 = 0$

Sol: $a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$

Order = 3

ch. Eqn: $r^3 - 2r^2 - r + 2 = 0$

$r = -1, 1, 2$

$a_n = \alpha_1(-1)^n + \alpha_2(1)^n + \alpha_3(2)^n$

$n = 0 \Rightarrow \alpha_1(-1)^0 + \alpha_2(1)^0 + \alpha_3(2)^0 = a_0$

$\alpha_1 + \alpha_2 + \alpha_3 = 0$

$n = 1 \Rightarrow \alpha_1(-1)^1 + \alpha_2(1)^1 + \alpha_3(2)^1 = a_1$

$-\alpha_1 + \alpha_2 + 2\alpha_3 = 6$

$n = 2 \Rightarrow \alpha_1(-1)^2 + \alpha_2(1)^2 + \alpha_3(2)^2 = a_2$

$\alpha_1 + \alpha_2 + 4\alpha_3 = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

$\alpha_1 = -3$, $\alpha_2 = 3$, $\alpha_3 = 0$

$a_0 = 3$

$a_1 = 6$

$a_2 = 0$

$a_n = -3(-1)^n + 3(1)^n + 0(2)^n$

$= -3(-1)^n + 3(1)^n$

4) $a_n + a_{n-1} - 6a_{n-2} = 0$ for $n \geq 2$ $a_0 = -1,\ a_1 = 8$

5) $a_n - 6a_{n-1} + 9a_{n-2} = 0$ for $n \geq 2$ $a_0 = 5,\ a_1 = 12$

4 Ans:) Given,

$$a_n + a_{n-1} - 6a_{n-2} = 0$$

Order = 2

Ch. Eqn: $r^2 - r - 6 = 0 \longrightarrow (1)$

$$r = \frac{+1 \pm \sqrt{1-4(-6)}}{2(1)} = \frac{1 \pm 5}{2}$$

$$r = 3, -2$$

$$a_n = \alpha_1 3^n - \alpha_2 2^n \longrightarrow (2)$$

$n = 0$ in (1) $\Rightarrow \alpha_1 3^0 - \alpha_2 2^0 = a_0$

$$\alpha_1 - \alpha_2 = -1$$

$$\alpha_1 = -1 + \alpha_2$$

$n = 1$ in (1) $\Rightarrow \alpha_1 \cdot 3^1 - \alpha_2 2^1 = a_1$

$$3\alpha_1 - 2\alpha_2 = 12$$

$$3(-1 + \alpha_2) - 2\alpha_2 = 12$$

$$-3 + 3\alpha_2 - 2\alpha_2 = 12$$

$\boxed{\alpha_2 = 15}$ $\boxed{\alpha_1 = 14}$

$$a_n = 14(3^n) - 15(2^n)$$

5 Ans: Given,

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

Order = 2

Ch. Eqn: $r^2 - 6r + 9 = 0 \longrightarrow (1)$

$$(r-3)^2 = 0$$

$$r = 3, 3$$

$$a_n = (\alpha_1 + \alpha_2 n)3^n \longrightarrow (2)$$

$n = 0$ in (2) $\Rightarrow$ $(\alpha_1 + \alpha_2 \cdot 0)3^0 = a_0$

$$\boxed{\alpha_1 = 5}$$

$n = 1$ in (2) $\Rightarrow$ $(\alpha_1 + \alpha_2 \cdot 1)3^1 = a_1$

$$(5 + \alpha_2)3 = 12$$

$$15 + 3\alpha_2 = 12$$

$$3\alpha_2 = 12 - 15$$

$$\boxed{\alpha_2 = -1}$$

α_1, α_2 in (2) | $a_n = (5 - n)3^n$

$a_0 = 5$ ✓

$a_1 = 12$ ✓

Solution of Non Homogeneous Recurrence Relation

* If recurrence relation is of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \dots + c_k a_{n-k} + G(n) \longrightarrow (1)$$

The general solution of this non-homogeneous recurrence relation

$$a_n = a_n^{(h)} + a_n^{(p)}$$

h – homogeneous sol^n
p – particular sol^n

Steps:

(i) We obtain the homogeneous solution

First we write associated homogeneous recurrence relation

$$f(n) = 0$$

$$(1) \Rightarrow \quad a_n + c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \dots + c_k a_{n-k} = 0$$

(ii) We obtain the particular solution

There is no general procedure for obtaining the particular sol. of a recurrence relation.

However, if $f(n)$ has any one of the following forms:

1.) Polynomial in 'n' $(2+3n)$ / $2n+(3n-1)$

2.) Constant $(2, 3, 4 \dots$ etc$)$

3.) Powers of constant $(2^n, 3^n \dots)$

Particular Solution for f(n)

S.No.	f(n)	Form of Particular Solution
1.)	Constant, C	Constant, d
2.)	Linear func^n, $c_0 + c_1 n$	$d_0 + d_1 k$
3.)	N	$d_0 + d_1 k$
4.)	n^2	$d_0 + d_1 k + d_2 k^2$
5.)	m^{th} degree polynomial $c_0 + c_1 n + c_2 n^2 + \cdots + c_m n^m$	$d_0 + d_1 k + d_2 k^2 + \cdots + d_m k^m$
6.)	r^n $r \in R$	$d r^n$

(iii) The general solution of the recurrence relation can be written as sum of homogeneous solution and all the particular solutions.

Q) Solve the following recurrence relation

$$a_n + 4a_{n-1} + 4a_{n-2} = 8 \text{ for } n \geq 2 \quad a_0 = 1, \; a_1 = 2$$

Sol: The given recurrence relation is:

$$a_n + 4a_{n-1} + 4a_{n-2} = 8 \longrightarrow (1)$$

It is a Non linear RR of 2nd order

The general solution of the reccurrence relation is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

(i) To obtain $a_n^{(h)}$, put $f(n) = 0$

$$(1) \Rightarrow a_n + 4a_{n-1} + 4a_{n-2} = 0$$

Characteristic Eqn: $r^2 + 4r + 4 = 0$

$$r = \frac{-4 \pm \sqrt{16 - 4(4)}}{2(1)} = \frac{-4}{2} = -2, -2$$

Roots are real & equal

general sol^n: $a_n^{(h)} = (\alpha_1 + \alpha_2 n)(-2)^n \longrightarrow (2)$

Put $n = 0$ in (2) $\Rightarrow (\alpha_1 + \alpha_2 \cdot 0)(-2)^0 = a_0$

$$\alpha_1 = 1$$

$n = 1 \Rightarrow (\alpha_1 + \alpha_2 \cdot 1)(-2)^1 = a_1$

$$(1 + \alpha_2)(-2) = 2$$

$$\alpha_2 = -2$$

$$a_n^{(h)} = (1-2n)(-2)^n$$

(ii) To obtain $a_n^{(p)} = d \longrightarrow$ (3) ($f(n) = A$ constant)

Put (3) in (1)

$$(1) \Rightarrow d + 4d + 4d = 8$$

$$9d = 8$$

$$\boxed{d = \frac{8}{9}}$$

(iii) The general solution is $a_n^{(h)} + a_n^{(p)}$

$$a_n = (\alpha_1 + \alpha_2 n)(-2)^n + \frac{8}{9} \longrightarrow (4)$$

Put $n=0$ in (4) $\Rightarrow (\alpha_1 + 0)(-2)^0 + \frac{8}{9} = a_0$

$$\alpha_1 + \frac{8}{9} = 1$$

$$\alpha_1 = 1 - \frac{8}{9} =$$

$$\boxed{\alpha_1 = \frac{1}{9}}$$

Put $n=1 \Rightarrow (\alpha_1 + \alpha_2)(-2) + \frac{8}{9} = 2$

$$-2\alpha_1 - 2\alpha_2 + \frac{8}{9} = 2$$

$$-2\left(\frac{1}{9}\right) - 2\alpha_2 + \frac{8}{9} = 2$$

$$-2\alpha_2 = 2 - \frac{2}{3} = \frac{4}{3} \qquad \boxed{\alpha_2 = \frac{-2}{3}}$$

Substituting α_1, α_2 in (4)

$$\therefore \quad a_n = \left(\frac{1}{9} - \frac{2}{3}n\right)(-2)^n + \frac{8}{9}$$

Q) $a_n - 2a_{n-1} + a_{n-2} = 2$ with $a_0 = 25$, $a_1 = 16$

Sol: Given,

$$a_n - 2a_{n-1} + a_{n-2} = 2 \longrightarrow (1)$$

It is a non-liner r.r of order 2

The general solution is $a_n = a_n^{(h)} + a_n^{(p)} \longrightarrow (2)$

(i) To obtain $a_n^{(h)}$, $f(n) = 0$

$$a_n - 2a_{n-1} + a_{n-2} = 0$$

Ch. Eq: $r^2 - 2r + 1 = 0$

$$(r-1)^2 = 0$$

$$r = 1, 1$$

$$a_n = (\alpha_1 + \alpha_2 n)\, 1^n$$

ii) To obtain $a_n^{(p)} = d$

$$d - 2d + d = \cancel{0}\,2$$

$$0 = 2 \quad \times$$

$$a_n^{(p)} = nd$$

$$nd - 2(n-1)d + (n-2)d = \cancel{0}\,2$$

$$\cancel{nd} - 2nd + 2\cancel{d} + \cancel{nd} - 2\cancel{d} = 2$$

Assume $\quad a_n^{(p)} = n^2 d$

$$n^2 d - 2(n-1)^2 d + (n-2)^2 d = 2$$

$$n^2 d - 2(n^2+1-2n)d + (n^2+4-4n)d = 2$$

$$\cancel{n^2 d} - \cancel{2n^2 d} + 2d + \cancel{4nd} + \cancel{n^2 d} + 4d - \cancel{4nd} = 2$$

$$2d = 2$$

$$\boxed{d = 1}$$

iii) $\quad a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = (\alpha_1 + \alpha_2 n) + n^2$$

$n = 0 \Rightarrow \quad (\alpha_1 + 0) + 0^2 = 25$

$$\alpha_1 = 25$$

$n = 1 \Rightarrow \quad (\alpha_1 + \alpha_2) + 1 = a_1$

$$25 + \alpha_2 + 1 = 16$$

$$\alpha_2 = 16 - 25 - 1$$

$$\alpha_2 = -10$$

$$a_n = (25 - 10n) + n^2$$

$$\therefore a_n = n^2 - 10n + 25$$

Q) Solve the following recurrence relation

$$a_n = 3a_{n-1} + 2^n \quad n \geq 1 \quad a_0 = 3$$

Sol: Given,

$$a_n = 3a_{n-1} + 2^n$$

$$\Rightarrow a_n - 3a_{n-1} = 2^n \longrightarrow (1)$$

The solution for the equation is:

$$a_n = a_n^{(h)} + a_n^{(p)} \longrightarrow (2)$$

Step 1: The associated homogeneous relation $a_n^{(h)}$

$$a_n - 3a_{n-1} = 0$$

Ch: Eqn $\Rightarrow$ $r - 3 = 0$

$$\boxed{r = 3}$$

$$a_n = \alpha_1 3^n \longrightarrow (3)$$

Step 2: To obtain particular solution of the given recurrence relation is

$$a_n^{(p)} = 2^n \longrightarrow (4)$$ and 2 is not the characteristic root

Let Particular sol^n be $a_n^{(P)} = d2^n \longrightarrow (5)$

sub. ⑤ in ①

$$\Rightarrow \quad d2^n - 3d\,2^{n-1} = 2^n$$

$$d\,\cancel{2^n}\left(1 - \frac{3}{2}\right) = \cancel{2^n}$$

$$d\left(\frac{-1}{3}\right) = 1$$

$$d = -3$$

$$d\,\cancel{2^{n-1}}(2-3) = \cancel{2^{n}}\,2$$

$\boxed{d = -2}$ in ⑤

$$a_n^{(p)} = (-2)2^n$$

Put values of $a_n^{(h)}$ & $a_n^{(p)}$ in ②

$$② \Rightarrow \quad a_n = \alpha 3^n + (-2)\,2^n$$

Put $n = 0 \Rightarrow$

$$\alpha 3^0 + (-2)\,2^0 = a_0$$

$$\alpha - 2 = 3$$

$$\boxed{\alpha = 5}$$

$$\therefore \; a_n = 5(3^n) - 2^{n+1}$$

Q) Solve the following recurrence relation

$$a_n = 2a_{n-1} + 2^n \quad n \geq 1 \quad a_0 = 2$$

Sol: Given,

$$a_n = 2a_{n-1} + 2^n$$

$$\Rightarrow a_n - 2a_{n-1} = 2^n \longrightarrow (1)$$

It is a non-linear RR with order 1

General sol^n is $a_n = a_n^{(h)} + a_n^{(p)} \longrightarrow (2)$

Step 1: Obtain $a_n^{(h)}$ by $f(n) = 0$

$$a_n - 2a_{n-1} = 0$$

ch. eqn is: $r - 2 = 0$

$$\boxed{r = 2}$$

$$a_n^{(h)} = \alpha 2^n \longrightarrow (3)$$

Step 2: Obtain $a_n^{(p)}$ by $a_n^{(p)} = 2^n$

$a_n^{(p)} = d2^n$ Characteristic root is equal

put $a_n^{(p)} = nd2^n$ in (1)

$$(1) \Rightarrow nd2^n - 2(n-1)d\,2^{n-1} = 2^n$$

$$\cancel{nd2^n} \quad \cancel{2^n}(nd - (n-1)d) = \cancel{2^n}$$

$\cancel{nd} - \cancel{nd} + d = 1$

$\boxed{d = 1}$

$a_n^{(p)} = n 2^n$

Put $a_n^{(h)}$ & $a_n^{(p)}$ values in (2)

(2) $\Rightarrow$ $a_n = \alpha 2^n + n 2^n$

$n = 0$ | $\alpha 2^0 + 0 = a_0$

$\alpha = 2$

$\therefore$ $a_n = 2^{n+1} + n 2^n$ (or) $2^n (2 + n)$

Q) $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$

Sol: $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n^{(h)}$: $r^2 + 5r + 6 = 0$

$r = \frac{-5 \pm \sqrt{25 - 24}}{2} = \frac{-5 \pm 1}{2} = -2, -3$

$a_n^{(h)} = \alpha_1 (-2)^n + \alpha_2 (-3)^n \longrightarrow (1)$

$$a_n^{(P)} = a_n^{(P_1)} * a_n^{(P_2)}$$

$$a_n^{(P_1)} = 3 \text{ (const.)} \qquad a_n^{(P_2)} = n^2$$

$$a_n^{(P_1)} = d \text{ in } 1 \qquad d_0 + d_1 k + d_2 k^2 = n^2$$

$$d + 5d + 6d = 3$$

$$12d = 3$$

$$d = \frac{1}{4}$$

$$a_n^{(P_2)} = n^2$$

$$d_0 + d_1 k + d_2 k^2 = n^2$$

$5a_{n-1}$

$$d_0 + d_1 n + d_2 n^2 + 5\left[d_0 + d_1(n-1) + d_2(n-1)^2\right] + 6\left[d_0 + d_1(n-2) + d_2(n-2)^2\right] = n^2$$

$$d_0 + d_1 n + d_2 n^2 + 5\left[d_0 + d_1 n - d_1 + d_2 n^2 + d_2 - \overset{n}{2}d_2\right] + 6\left[d_0 + d_1 n - 2d_1 + n^2 d_2 + 4d_2 - 4nd_2\right] = n^2$$

$$d_0 + nd_1 + n^2 d_2 + 5d_0 + \overset{5}{}nd_1 - 5d_1 + 5n^2 d_2 + 5d_2 - 10nd_2 + 6d_0$$

$$+ 6nd_1 - 12d_1 + 6n^2 d_2 + 24d_2 - 24nd_2 = n^2$$

$$12d_0 + 8nd_1 + 12n^2 d_2 + 17d_1 + 29d_2 - 34nd_2 = n^2$$

$$12n^2 d_2 - 34nd_2 + \overset{12}{8}nd_1 + 12d_0 + 17d_1 + 29d_2 = n^2$$

$$(12d_2 - 1)n^2 + n(12d_1 - 34d_2) + (12d_0 - 17d_1 + 29d_2) = 0$$

$12d_2 - 1 = 0$

$\boxed{d_2 = \frac{1}{12}}$

$12d_1 = 34d_2$

$d_1 = \frac{34}{12}\left(\frac{1}{12}\right)$

$\boxed{d_1 = \frac{17}{72}}$

$12d_0 - 17d_1 + 29d_2 = 0$

$d_0 = \frac{1}{12}\left(17\left(\frac{17}{72}\right) - 29\left(\frac{1}{12}\right)\right)$

$d_0 = \frac{1}{12}\left(\frac{289}{72} - \frac{29}{12}\right)$

Compare coefficients

$\boxed{d_0 = \frac{115}{864}}$

$$a_n^{(P_2)} = \left(\frac{115}{864} + \frac{17}{72}n + \frac{1}{12}n^2\right) \times \frac{1}{4}$$

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$a_n = \alpha_1(-2)^n + \alpha_2(-3)^n + \frac{1}{4}\left(\frac{115}{864} + \frac{17n}{72} + \frac{n^2}{12}\right)$$