Unit – I

Mathematical Logic

INTRODUCTION

Proposition: A **proposition** or **statement** is a declarative sentence which is either true or false but not both. The truth or falsity of a proposition is called its **truth-value.**

These two values 'true' and 'false' are denoted by the symbols T and F respectively. Sometimes these are also denoted by the symbols 1 and 0 respectively.

Example 1: Consider the following sentences:

1. Delhi is the capital of India.
2. Kolkata is a country.
3. 5 is a prime number.
4. 2 + 3 = 4.

These are propositions (or statements) because they are either true of false.

Next consider the following sentences:

5. How beautiful are you?
6. Wish you a happy new year
7. $x + y = z$
8. Take one book.

These are not propositions as they are not declarative in nature, that is, they do not declare a definite truth value T or F.

Propositional Calculus is also known as **statement calculus.** It is the branch of mathematics that is used to describe a logical system or structure. A logical system consists of (1) a universe of propositions, (2) truth tables (as axioms) for the logical operators and (3) definitions that explain equivalence and implication of propositions.

Connectives

The words or phrases or symbols which are used to make a proposition by two or more propositions are called **logical connectives** or **simply connectives.** There are five basic connectives called negation, conjunction, disjunction, conditional and biconditional.

Negation

The **negation** of a statement is generally formed by writing the word 'not' at a proper place in the statement (proposition) or by prefixing the statement with the phrase 'It is not the case that'. If p denotes a statement then the negation of p is written as p and read as 'not p'. If the truth value of p is T then the truth value of p is F. Also if the truth value of p is F then the truth value of p is T.

Table 1. Truth table for negation

p	¬p
T	F
F	T

Example 2: Consider the statement p: Kolkata is a city. Then $\neg p$: Kolkata is not a city.

Although the two statements 'Kolkata is not a city' and 'It is not the case that Kolkata is a city' are not identical, we have translated both of them by p. The reason is that both these statements have the same meaning.

Conjunction

The **conjunction** of two statements (or propositions) p and q is the statement $p \wedge q$ which is read as 'p and q'. The statement $p \wedge q$ has the truth value T whenever both p and q have the truth value T. Otherwise it has truth value F.

Table 2. Truth table for conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 3: Consider the following statements p : It is raining today.

q : There are 10 chairs in the room.

Then $p \wedge q$: It is raining today and there are 10 chairs in the room.

Note: Usually, in our everyday language the conjunction 'and' is used between two statements which have some kind of relation. Thus a statement 'It is raining today and $1 + 1 = 2$' sounds odd, but in logic it is a perfectly acceptable statement formed from the statements 'It is raining today' and '$1 + 1 = 2$'.

Example 4: Translate the following statement:

'Jack and Jill went up the hill' into symbolic form using conjunction.

Solution: Let p : Jack went up the hill, q : Jill went up the hill.

Then the given statement can be written in symbolic form as $p \wedge q$.

Disjunction

The **disjunction** of two statements p and q is the statement $p \vee q$ which is read as 'p or q'. The statement $p \vee q$ has the truth value F only when both p and q have the truth value F. Otherwise it has truth value T.

Table 3: Truth table for disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 5: Consider the following statements p : I shall go to the game.

q : I shall watch the game on television.

Then $p \vee q$: I shall go to the game or watch the game on television.

Conditional proposition

If p and q are any two statements (or propositions) then the statement $p \rightarrow q$ which is read as, 'If p, then q' is called a **conditional statement** (or **proposition**) or **implication** and the connective is the **conditional connective.**

The conditional is defined by the following table:

Table 4. Truth table for conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In this conditional statement, p is called the **hypothesis** or **premise** or **antecedent** and q is called the **consequence** or **conclusion.**

To understand better, this connective can be looked as a conditional promise. If the promise is violated (broken), the conditional (implication) is false. Otherwise it is true. For this reason, the only circumstances under which the conditional $p \rightarrow q$ is false is when p is true and q is false.

Example 6: *Translate the following statement:*

'The crop will be destroyed if there is a flood' into symbolic form using conditional connective.

Solution: Let c : the crop will be destroyed; f : there is a flood.
Let us rewrite the given statement as
'If there is a flood, then the crop will be destroyed'. So, the symbolic form of the given statement is $f \rightarrow c$.

Example 7: Let p and q denote the statements:
p : You drive over 70 km per hour.
q : You get a speeding ticket.

Write the following statements into symbolic forms.

(i) You will get a speeding ticket if you drive over 70 km per hour.

(ii) Driving over 70 km per hour is sufficient for getting a speeding ticket.

(iii) If you do not drive over 70 km per hour then you will not get a speeding ticket.
(iv) Whenever you get a speeding ticket, you drive over 70 km per hour.
Solution: (i) $p \rightarrow q$ (ii) $p \rightarrow q$ (iii) $p \rightarrow q$ (iv) $q \rightarrow p$.

Notes: 1. In ordinary language, it is customary to assume some kind of relationship between the antecedent and the consequent in using the conditional. But in logic, the antecedent and the consequent in a conditional statement are not required to refer to the same subject matter. For example, the statement ‘If I get sufficient money then I shall purchase a high-speed computer’ sounds reasonable. On the other hand, a statement such as ‘If I purchase a computer then this pen is red’ does not make sense in our conventional language. But according to the definition of conditional, this proposition is perfectly acceptable and has a truth-value which depends on the truth-values of the component statements.

2. Some of the alternative terminologies used to express $p \to q$ (if p, then q) are the following: (*i*) p implies q

(*ii*) p only if q (‘If p, then q’ formulation emphasizes the antecedent, whereas ‘p only if q’ formulation emphasizes the consequent. The difference is only stylistic.)

(*iii*) q if p, or q when p.

(*iv*) q follows from p, or q whenever p.

(*v*) p is sufficient for q, or a sufficient condition for q is p. (*vi*) q is necessary for p, or a necessary condition for p is q. (*vii*) q is consequence of p.

Converse, Inverse and Contrapositive

If $P \to Q$ is a conditional statement, then

(1). $Q \to P$ is called its *converse*

(2). $\neg P \to \neg Q$ is called its *inverse*

(3). $\neg Q \to \neg P$ is called its *contrapositive*.

Truth table for $Q \to P$ (converse of $P \to Q$)

P	Q	$Q \to P$
T	T	T
T	F	T
F	T	F
F	F	T

Truth table for $\neg P \to \neg Q$ (inverse of $P \to Q$)

P	Q	$\neg P$	$\neg Q$	$\neg P \to \neg Q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Truth table for $\neg Q \to \neg P$ (contrapositive of $P \to Q$)

P	Q	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Example: Consider the statement

P : It rains.

Q: The crop will grow.

The implication $P \rightarrow Q$ states that

R: If it rains then the crop will grow.

The converse of the implication $P \rightarrow Q$, namely $Q \rightarrow P$ sates that S: If the crop will grow then there has been rain.

The inverse of the implication $P \rightarrow Q$, namely $\neg P \rightarrow \neg Q$ sates that

U: If it does not rain then the crop will not grow.

The contraposition of the implication $P \rightarrow Q$, namely $\neg Q \rightarrow \neg P$ states that T : If the crop do not grow then there has been no rain.

Example 9: *Construct the truth table for* $(p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Biconditional proposition

If p and q are any two statements (propositions), then the statement $p \leftrightarrow q$ which is read as 'p if and only if q' and abbreviated as 'p iff q' is called a **biconditional statement** and the connective is the **biconditional connective.**

The truth table of $p \leftrightarrow q$ is given by the following table:

Table 6. Truth table for biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

It may be noted that p q is true only when both p and q are true or when both p and q are false. Observe that p q is true when both the conditionals $p \rightarrow q$ and $q \rightarrow p$ are true, *i.e.*, the truth-values of $(p \rightarrow q) \wedge (q \rightarrow p)$, given in Ex. 9, are identical to the truth-values of p q defined here.

Note: The notation $p \leftrightarrow q$ is also used instead of $p \leftrightarrow q$.

TAUTOLOGY AND CONTRADICTION

Tautology: A statement formula which is true regardless of the truth values of the statements which replace the variables in it is called a **universally valid formula** or a **logical truth** or a **tautology.**

Contradiction: A statement formula which is false regardless of the truth values of the statements which replace the variables in it is said to be a **contradiction.**

Contingency: A statement formula which is neither a tautology nor a contradiction is known as a **contingency.**

Substitution Instance

A formula A is called a substitution instance of another formula B if A can be obtained form B by substituting formulas for some variables of B, with the condition that the same formula is substituted for the same variable each time it occurs.

Example: Let $B : P \to (J \wedge P)$.

Substitute $R \leftrightarrow S$ for P in B, we get

$$(i): (R \leftrightarrow S) \to (J \wedge (R \leftrightarrow S))$$

Then A is a substitution instance of B.

Note that $(R \leftrightarrow S) \to (J \wedge P)$ is not a substitution instance of B because the variables

P in $J \wedge P$ was not replaced by $R \leftrightarrow S$.

Equivalence of Formulas

Two formulas A and B are said to equivalent to each other if and only if $A \leftrightarrow B$ is a tautology.

If $A \leftrightarrow B$ is a tautology, we write $A \Leftrightarrow B$ which is read as A is equivalent to B.

Note : 1. $\Leftrightarrow$ is only symbol, but not connective.

2. $A \leftrightarrow B$ is a tautology if and only if truth tables of A and B are the same.
3. Equivalence relation is symmetric and transitive.

Method I. Truth Table Method: One method to determine whether any two statement formulas are equivalent is to construct their truth tables.

Example: Prove $P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$.

Solution:

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$	$(P \vee Q) \Leftrightarrow \neg(\neg P \wedge \neg Q)$
T	T	T	F	F	F	T	T
T	F	T	F	T	F	T	T
F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T

As $P \vee Q$ $\quad$ $\neg(\neg P \wedge \neg Q)$ is a tautology, then $P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$.

Example: Prove $(P \to Q) \Leftrightarrow (\neg P \vee Q)$.

Solution:

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$	$(P \to Q)$ $\quad$ $(\neg P \vee Q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As $(P \to Q)$ $\quad$ $(\neg P \vee Q)$ is a tautology then $(P \to Q) \Leftrightarrow (\neg P \vee Q)$.

Equivalence Formulas:

1. Idempotent laws:

   (a) $P \vee P \Leftrightarrow P$ (b) $P \wedge P \Leftrightarrow P$

2. Associative laws:

   (a) $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ (b) $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

3. Commutative laws:

   (a) $P \vee Q \Leftrightarrow Q \vee P$ (b) $P \wedge Q \Leftrightarrow Q \wedge P$

4. Distributive laws:

   $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

5. Identity laws:

   (a) (i) $P \vee F \Leftrightarrow P$ (ii) $P \vee T \Leftrightarrow T$

   (b) (i) $P \wedge T \Leftrightarrow P$ (ii) $P \wedge F \Leftrightarrow F$

6. Component laws:

   (a) (i) $P \vee \neg P \Leftrightarrow T$ (ii) $P \wedge \neg P \Leftrightarrow F$.

   (b) (i) $\neg\neg P \Leftrightarrow P$ (ii) $\neg T \Leftrightarrow F$, $\neg F \Leftrightarrow T$

7. Absorption laws:

   (a) $P \vee (P \wedge Q) \Leftrightarrow P$ (b) $P \wedge (P \vee Q) \Leftrightarrow P$

8. Demorgan's laws:

   (a) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ (b) $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

Method II. Replacement Process: Consider a formula $A : P \rightarrow (Q \rightarrow R)$. The formula $Q \rightarrow R$ is a part of the formula A. If we replace $Q \rightarrow R$ by an equivalent formula $\neg Q \vee R$ in A, we get another formula $B : P \rightarrow (\neg Q \vee R)$. One can easily verify that the formulas A and B are equivalent to each other. This process of obtaining B from A as the replacement process.

Example: Prove that $P \rightarrow (Q \rightarrow R) \Leftrightarrow P \rightarrow (\neg Q \vee R) \Leftrightarrow (P \wedge Q) \rightarrow R$.(May. 2010)

Solution: $P \rightarrow (Q \rightarrow R) \Leftrightarrow P \rightarrow (\neg Q \vee R)$ $[\because Q \rightarrow R \Leftrightarrow \neg Q \vee R]$

$\Leftrightarrow \neg P \vee (\neg Q \vee R)$ $[\because P \rightarrow Q \Leftrightarrow \neg P \vee Q]$

$\Leftrightarrow (\neg P \vee \neg Q) \vee R$ [by Associative laws]

$\Leftrightarrow \neg(P \wedge Q) \vee R$ [by De Morgan's laws]

$\Leftrightarrow (P \wedge Q) \rightarrow R[\because P \rightarrow Q \Leftrightarrow \neg P \vee Q]$.

Example: Prove that $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (P \vee R) \rightarrow Q$.

Solution: $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q)$

$\Leftrightarrow (\neg P \wedge \neg R) \vee Q \Leftrightarrow$

$\neg(P \vee R) \vee Q \Leftrightarrow P \vee$

$R \rightarrow Q$

Example: Prove that $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$.

Solution: $P \to (Q \to P) \Leftrightarrow \neg P \vee (Q \to P)$

$\Leftrightarrow \neg P \vee (\neg Q \vee P)$

$\Leftrightarrow (\neg P \vee P) \vee \neg Q$

$\Leftrightarrow T \vee \neg Q$

$\Leftrightarrow T$

and

$\neg P \to (P \to Q) \Leftrightarrow \neg(\neg P) \vee (P \to Q)$

$\Leftrightarrow P \vee (\neg P \vee Q) \Leftrightarrow$

$(P \vee \neg P) \vee Q \Leftrightarrow T$

$\vee Q$

$\Leftrightarrow T$

So, $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$.

***Example: Prove that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$. (Nov. 2009)

Solution:

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$ [Associative and Distributive laws]

$\Leftrightarrow (\neg(P \vee Q) \wedge R) \vee ((Q \vee P) \wedge R)$ [De Morgan's laws]

$\Leftrightarrow (\neg(P \vee Q) \vee (P \vee Q)) \wedge R$ [Distributive laws]

$\Leftrightarrow T \wedge R$ [$\because \neg P \vee P \Leftrightarrow T$]

$\Leftrightarrow R$

**Example: Show $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is tautology.

Solution: By De Morgan's laws, we have

$$\neg P \wedge \neg Q \Leftrightarrow \neg(P \vee Q)$$

$$\neg P \vee \neg R \Leftrightarrow \neg(P \wedge R)$$

Therefore

$$(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R) \Leftrightarrow \neg(P \vee Q) \vee \neg(P \wedge R)$$

$$\Leftrightarrow \neg((P \vee Q) \wedge (P \vee R))$$

Also

$$\neg(\neg P \wedge (\neg Q \vee \neg R)) \Leftrightarrow \neg(\neg P \wedge \neg(Q \wedge R))$$

$$\Leftrightarrow P \vee (Q \wedge R)$$

$$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

Hence $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \Leftrightarrow (P \vee Q) \wedge (P \vee Q) \wedge (P \vee R)$

$$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

Thus $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$

$\Leftrightarrow [(P \vee Q) \wedge (P \vee R)] \vee \neg[(P \vee Q) \wedge (P \vee R)]$

$\Leftrightarrow T$

Hence the given formula is a tautology.

Example: Show that $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology. (Nov. 2009)

Solution: $(P \wedge Q) \rightarrow (P \vee Q) \Leftrightarrow \neg(P \wedge Q) \vee (P \vee Q)$ [$\because P \rightarrow Q \Leftrightarrow \neg P \vee Q$]

$\Leftrightarrow (\neg P \vee \neg Q) \vee (P \vee Q)$ [by De Morgan's laws]

$\Leftrightarrow (\neg P \vee P) \vee (\neg Q \vee Q)$ [by Associative laws and commutative laws]

$\Leftrightarrow (T \vee T)$[by negation laws]

$\Leftrightarrow T$

Hence, the result.

Example: Write the negation of the following statements.

(a). Jan will take a job in industry or go to graduate school.
(b). James will bicycle or run tomorrow.
(c). If the processor is fast then the printer is slow.

Solution: (a). Let P : Jan will take a job in industry.
Q: Jan will go to graduate school.

The given statement can be written in the symbolic as $P \vee Q$.

The negation of $P \vee Q$ is given by $\neg(P \vee Q)$.

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q.$$

$\neg P \wedge \neg Q$: Jan will not take a job in industry and he will not go to graduate school.

(b). Let P : James will bicycle.
Q: James will run tomorrow.

The given statement can be written in the symbolic as $P \vee Q$.

The negation of $P \vee Q$ is given by $\neg(P \vee Q)$.

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q.$$

$\neg P \wedge \neg Q$: James will not bicycle and he will not run tomorrow.

(c). Let P : The processor is fast.
Q: The printer is slow.

The given statement can be written in the symbolic as $P \rightarrow Q$.

The negation of $P \rightarrow Q$ is given by $\neg(P \rightarrow Q)$.

$$\neg(P \rightarrow Q) \Leftrightarrow \neg(\neg P \vee Q) \Leftrightarrow P \wedge \neg Q.$$

$P \wedge \neg Q$: The processor is fast and the printer is fast.

Example: Use Demorgans laws to write the negation of each statement.

(a). I want a car and worth a cycle.
(b). My cat stays outside or it makes a mess.
(c). I've fallen and I can't get up.
(d). You study or you don't get a good grade.

Solution: (a). I don't want a car or not worth a cycle.
(b). My cat not stays outside and it does not make a mess.

(c). I have not fallen or I can get up.

(d). You can not study and you get a good grade.

Exercises: 1. Write the negation of the following statements.

(a). If it is raining, then the game is canceled.

(b). If he studies then he will pass the examination.

Are $(p \to q) \to r$ and $p \to (q \to r)$ logically equivalent? Justify your answer by using the rules of logic to simply both expressions and also by using truth tables. Solution: $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent because

Method I: Consider

$$(p \to q) \to r \Leftrightarrow (\neg p \lor q) \to r$$

$$\Leftrightarrow \neg(\neg p \lor q) \lor r \Leftrightarrow$$

$$(p \land \neg q) \lor r$$

$$\Leftrightarrow (p \land r) \lor (\neg q \land r)$$

and

$$p \to (q \to r) \Leftrightarrow p \to (\neg q \lor r)$$

$$\Leftrightarrow \neg p \lor (\neg q \lor r) \Leftrightarrow$$

$$\neg p \lor \neg q \lor r.$$

Method II: (Truth Table Method)

p	q	r	$p \to q$	$(p \to q) \to r$		$q \to r$	$p \to (q \to r)$
T	T	T	T		T	T	T
T	T	F	T		F	F	F
T	F	T	F		T	T	T
T	F	F	F		T	T	T
F	T	T	T		T	T	T
F	T	F	T		F	F	T
F	F	T	T		T	T	T
F	F	F	T		F	T	T

Here the truth values (columns) of $(p \to q) \to r$ and $p \to (q \to r)$ are not identical.

Consider the statement: ”If you study hard, then you will excel”. Write its converse, contra positive and logical negation in logic.

Duality Law

Two formulas A and A^* are said to be *duals* of each other if either one can be obtained from the other by replacing $\land$ by $\lor$ and $\lor$ by $\land$. The connectives $\lor$ and $\land$ are called *duals* of each other. If the formula A contains the special variable T or F, then A^*, its dual is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

Example: Write the dual of the following formulas:

(i). $(P \vee Q) \wedge R$ (ii). $(P \wedge Q) \vee T$ (iii). $(P \wedge Q) \vee (P \vee \neg(Q \wedge \neg S))$

Solution: The duals of the formulas may be written as

(i). $(P \wedge Q) \vee R$ (ii). $(P \vee Q) \wedge F$ (iii). $(P \vee Q) \wedge (P \wedge \neg(Q \vee \neg S))$

Result 1: The negation of the formula is equivalent to its dual in which every variable is replaced by its negation.

We can prove

$$\neg A(P_1, P_2, ..., P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2, ..., \neg P_n)$$

Example: Prove that (a). $\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (\neg P \vee Q)$

(b). $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$

Solution: (a).$\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (P \wedge Q) \vee (\neg P \vee (\neg P \vee Q))$ $[\because P \rightarrow Q \Leftrightarrow \neg P \vee Q]$

$$\Leftrightarrow (P \wedge Q) \vee (\neg P \vee Q)$$

$$\Leftrightarrow (P \wedge Q) \vee \neg P \vee Q$$

$$\Leftrightarrow ((P \wedge Q) \vee \neg P)) \vee Q$$

$$\Leftrightarrow ((P \vee \neg P) \wedge (Q \vee \neg P)) \vee Q$$

$$\Leftrightarrow (T \wedge (Q \vee \neg P)) \vee Q$$

$$\Leftrightarrow (Q \vee \neg P) \vee Q$$

$$\Leftrightarrow Q \vee \neg P$$

$$\Leftrightarrow \neg P \vee Q$$

(b). From (a)

$$(P \wedge Q) \vee (\neg P \vee (\neg P \vee Q)) \Leftrightarrow \neg P \vee Q$$

Writing the dual

$$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$$

Tautological Implications

A statement formula A is said to *tautologically imply* a statement B if and only if $A \rightarrow B$ is a tautology.

In this case we write $A \Rightarrow B$, which is read as 'A implies B'.

Note: $\Rightarrow$ is not a connective, $A \Rightarrow B$ is not a statement formula.

$A \Rightarrow B$ states that $A \rightarrow B$ is tautology.

Clearly $A \Rightarrow B$ guarantees that B has a truth value T whenever A has the truth value T.

One can determine whether $A \Rightarrow B$ by constructing the truth tables of A and B in the same manner as was done in the determination of $A \Leftrightarrow B$. Example: Prove that $(P \rightarrow Q) \Rightarrow (\neg Q \rightarrow \neg P)$.

Solution:

P	Q	$\neg P$	$\neg Q$	$P \rightarrow Q$	$\neg Q \rightarrow \neg P$	$(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since all the entries in the last column are true, $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$ is a tautology.

Hence $(P \rightarrow Q) \Rightarrow (\neg Q \rightarrow \neg P)$.

In order to show any of the given implications, it is sufficient to show that an assignment of the truth value T to the antecedent of the corresponding condi-

tional leads to the truth value T for the consequent. This procedure guarantees that the conditional becomes tautology, thereby proving the implication.

Example: Prove that $\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$.

Solution: Assume that the antecedent $\neg Q \wedge (P \rightarrow Q)$ has the truth value T, then both $\neg Q$ and $P \rightarrow Q$ have the truth value T, which means that Q has the truth value F, $P \rightarrow Q$ has the truth value T. Hence P must have the truth value F.

Therefore the consequent $\neg P$ must have the truth value T.

$$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P.$$

Another method to show $A \Rightarrow B$ is to assume that the consequent B has the truth value F and then show that this assumption leads to A having the truth value F. Then $A \rightarrow B$ must have the truth value T.

Example: Show that $\neg(P \rightarrow Q) \Rightarrow P$.

Solution: Assume that P has the truth value F. When P has F, $P \rightarrow Q$ has T, then $\neg(P \rightarrow Q)$ has F. Hence $\neg(P \rightarrow Q) \rightarrow P$ has T.

$$\neg(P \rightarrow Q) \Rightarrow P$$

Other Connectives

We introduce the connectives NAND, NOR which have useful applications in the design of computers.

NAND: The word NAND is a combination of 'NOT' and 'AND' where 'NOT' stands for negation and 'AND' for the conjunction. It is denoted by the symbol $\uparrow$.

If P and Q are two formulas then

$$P \uparrow Q \Leftrightarrow \neg(P \wedge Q)$$

The connective $\uparrow$ has the following equivalence:

$$P \uparrow P \Leftrightarrow \neg(P \wedge P) \Leftrightarrow \neg P \vee \neg P \Leftrightarrow \neg P.$$

$$(P \uparrow Q) \uparrow (P \uparrow Q) \Leftrightarrow \neg(P \uparrow Q) \Leftrightarrow \neg(\neg(P \wedge Q)) \Leftrightarrow P \wedge Q.$$

$$(P \uparrow P) \uparrow (Q \uparrow Q) \Leftrightarrow \neg P \uparrow \neg Q \Leftrightarrow \neg(\neg P \wedge \neg Q) \Leftrightarrow P \vee Q.$$

NAND is Commutative: Let P and Q be any two statement formulas.

$$(P \uparrow Q) \Leftrightarrow \neg(P \wedge Q)$$

$$\Leftrightarrow \neg(Q \wedge P) \Leftrightarrow$$

$$(Q \uparrow P)$$

$\therefore$ NAND is commutative.

NAND is not Associative: Let P, Q and R be any three statement formulas.

Consider $\uparrow (Q \uparrow R) \Leftrightarrow \neg(P \wedge (Q \uparrow R)) \Leftrightarrow \neg(P \wedge (\neg(Q \wedge R)))$

$$\Leftrightarrow \neg P \vee (Q \wedge R))$$

$$(P \uparrow Q) \uparrow R \Leftrightarrow \neg(P \wedge Q) \uparrow R$$

$$\Leftrightarrow \neg(\neg(P \wedge Q) \wedge R) \Leftrightarrow$$

$$(P \wedge Q) \vee \neg R$$

Therefore the connective $\uparrow$ is not associative.

NOR: The word NOR is a combination of 'NOT' and 'OR' where 'NOT' stands for negation and 'OR' for the disjunction. It is denoted by the symbol $\downarrow$.

If P and Q are two formulas then

$$P \downarrow Q \Leftrightarrow \neg(P \vee Q)$$

The connective $\downarrow$ has the following equivalence:

$$P \downarrow P \Leftrightarrow \neg(P \vee P) \Leftrightarrow \neg P \wedge \neg P \Leftrightarrow \neg P.$$

$$(P \downarrow Q) \downarrow (P \downarrow Q) \Leftrightarrow \neg(P \downarrow Q) \Leftrightarrow \neg(\neg(P \vee Q)) \Leftrightarrow P \vee Q.$$

$$(P \downarrow P) \downarrow (Q \downarrow Q) \Leftrightarrow \neg P \downarrow \neg Q \Leftrightarrow \neg(\neg P \vee \neg Q) \Leftrightarrow P \wedge Q.$$

NOR is Commutative: Let P and Q be any two statement formulas.

$$(P \downarrow Q) \Leftrightarrow \neg(P \vee Q)$$

$$\Leftrightarrow \neg(Q \vee P) \Leftrightarrow$$

$$(Q \downarrow P)$$

$\therefore$ NOR is commutative.

NOR is not Associative: Let P, Q and R be any three statement formulas. Consider

$$P \downarrow (Q \downarrow R) \Leftrightarrow \neg(P \vee (Q \downarrow R))$$

$$\Leftrightarrow \neg(P \vee (\neg(Q \vee R)))$$

$$\Leftrightarrow \neg P \wedge (Q \vee R)$$

$$(P \downarrow Q) \downarrow R \Leftrightarrow \neg(P \vee Q) \downarrow R$$

$$\Leftrightarrow \neg(\neg(P \vee Q) \vee R) \Leftrightarrow$$

$$(P \vee Q) \wedge \neg R$$

Therefore the connective $\downarrow$ is not associative.

Evidently, $P \uparrow Q$ and $P \downarrow Q$ are duals of each other.

Since

$$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$$

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q.$$

Example: Express $P \downarrow Q$ interms of $\uparrow$ only.
Solution:

$$\begin{aligned} \downarrow Q &\Leftrightarrow \neg(P \vee Q) \\ &\Leftrightarrow (P \vee Q) \uparrow (P \vee Q) \\ &\Leftrightarrow [(P \uparrow P) \uparrow (Q \uparrow Q)] \uparrow [(P \uparrow P) \uparrow (Q \uparrow Q)] \end{aligned}$$

Example: Express $P \uparrow Q$ interms of $\downarrow$ only. (May-2012)

Solution:

$$\begin{aligned} \uparrow Q &\Leftrightarrow \neg(P \wedge Q) \\ &\Leftrightarrow (P \wedge Q) \downarrow (P \wedge Q) \\ &\Leftrightarrow [(P \downarrow P) \downarrow (Q \downarrow Q)] \downarrow [(P \downarrow P) \downarrow (Q \downarrow Q)] \end{aligned}$$

Truth Tables

Example: Show that $(A \oplus B) \vee (A \downarrow B) \Leftrightarrow (A \uparrow B)$. (May-2012)
Solution: We prove this by constructing truth table.

A	B	$A \oplus B$	$A \downarrow B$	$(A \oplus B) \vee (A \downarrow B)$	$A \uparrow B$
T	T	F	F	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	T	T	T

As columns $(A \oplus B) \vee (A \downarrow B)$ and $(A \uparrow B)$ are identical.

$\therefore (A \oplus B) \vee (A \downarrow B) \Leftrightarrow (A \uparrow B)$.

Normal Forms

If a given statement formula $A(p_1, p_2, ...p_n)$ involves n atomic variables, we have 2^n possible combinations of truth values of statements replacing the variables.

The formula A is a tautology if A has the truth value T for all possible assignments of the truth values to the variables $p_1, p_2, ...p_n$ and A is called a contradiction if A has the truth value F for all possible assignments of the truth values of the n variables. A is said to be *satis able* if A has the truth value T for atleast one combination of truth values assigned to $p_1, p_2, ...p_n$.

The problem of determining whether a given statement formula is a Tautology, or a Contradiction is called a decision problem.

The construction of truth table involves a finite number of steps, but the construc-tion may not be practical. We therefore reduce the given statement formula to normal form and find whether a given statement formula is a Tautology or Contradiction or atleast satisfiable.

It will be convenient to use the word ”product” in place of ”conjunction” and ”sum” in place of ”disjunction” in our current discussion.

A product of the variables and their negations in a formula is called an *elementary product*. Similarly, a sum of the variables and their negations in a formula is called an *elementary sum*.

Let P and Q be any atomic variables. Then P , $\neg P \wedge Q$, $\neg Q \wedge P \;\; \neg P$, $P \;\; \neg P$, and $Q \wedge \neg P$ are some examples of elementary products. On the other hand, P , $\neg P \vee Q$, $\neg Q \vee P \vee \neg P$, $P \vee \neg P$, and $Q \vee \neg P$ are some examples of elementary sums.

Any part of an elementary sum or product which is itself an elementary sum or product is called a *factor* of the original elementary sum or product. Thus $\neg Q, \wedge \neg P$, and $\neg Q \wedge P$ are some of the factors of $\neg Q \wedge P \wedge \neg P$.

Disjunctive Normal Form (DNF)

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a *disjunctive normal form* of the given formula.

Example: Obtain disjunctive normal forms of

(a) $P \wedge (P \rightarrow Q)$; (b) $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$.

Solution: (a) We have

$$P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge (\neg P \vee Q)$$
$$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$$

(b) $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$

$$\Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q)) \text{ [using}$$
$$R \leftrightarrow S \Leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S)$$
$$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$$
$$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q)$$
$$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$$

which is the required disjunctive normal form.

Note: The DNF of a given formula is not unique.

Conjunctive Normal Form (CNF)

A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a *conjunctive normal form* of the given formula.

The method for obtaining conjunctive normal form of a given formula is similar to the one given for disjunctive normal form. Again, the conjunctive normal form is not unique.

Example: Obtain conjunctive normal forms of

(a) $P \wedge (P \rightarrow Q)$; (b) $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$.

Solution: (a). $P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge (\neg P \vee Q)$

(b). $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$

$$\Leftrightarrow (\neg(P \vee Q) \rightarrow (P \wedge Q)) \wedge ((P \wedge Q) \rightarrow \neg(P \vee Q))$$

$$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee \neg(P \vee Q))$$

$$\Leftrightarrow [(P \vee Q \vee P) \wedge (P \vee Q \vee Q)] \wedge [(\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q)]$$

$$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q)$$

Note: A given formula is tautology if every elementary sum in CNF is tautology.

Example: Show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

Solution: First we obtain a CNF of the given formula.

$$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) \Leftrightarrow Q \vee ((P \vee \neg P) \wedge \neg Q)$$

$$\Leftrightarrow (Q \vee (P \vee \neg P)) \wedge (Q \vee \neg Q)$$

$$\Leftrightarrow (Q \vee P \vee \neg P) \wedge (Q \vee \neg Q)$$

Since each of the elementary sum is a tautology, hence the given formula is tautology.

Principal Disjunctive Normal Form

In this section, we will discuss the concept of principal disjunctive normal form (PDNF).

Minterm: For a given number of variables, the minterm consists of conjunctions in which each statement variable or its negation, but not both, appears only once.

Let P and Q be the two statement variables. Then there are 2^2 minterms given by $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, and $\neg P \wedge \neg Q$.

Minterms for three variables P, Q and R are $P \wedge Q \wedge R$, $P \wedge Q \wedge \neg R$, $P \wedge \neg Q \wedge R$, $P \wedge \neg Q \wedge \neg R$, $\neg P \wedge Q \wedge R$, $\neg P \wedge Q \wedge \neg R$, $\neg P \wedge \neg Q \wedge R$ and $\neg P \wedge \neg Q \wedge \neg R$. From the truth tables of these minterms of P and Q, it is clear that

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

(i). no two minterms are equivalent

(ii). Each minterm has the truth value T for exactly one combination of the truth values of the variables P and Q.

Definition: For a given formula, an equivalent formula consisting of disjunctions of minterms only is called the Principal disjunctive normal form of the formula.

The principle disjunctive normal formula is also called the sum-of-products canonical form.

Methods to obtain PDNF of a given formula

(a). By Truth table:

(i). Construct a truth table of the given formula.

(ii). For every truth value T in the truth table of the given formula, select the minterm which also has the value T for the same combination of the truth values of P and Q.

(iii). The disjunction of these minterms will then be equivalent to the given formula.

Example: Obtain the PDNF of $P \to Q$.
Solution: From the truth table of $P \to Q$

P	Q	$P \to Q$	Minterm
T	T	T	$P \wedge Q$
T	F	F	$P \wedge \neg Q$
F	T	T	$\neg P \wedge Q$
F	F	T	$\neg P \wedge \neg Q$

The PDNF of $P \to Q$ is $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$.

$$\therefore P \to Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q).$$

Example: Obtain the PDNF for $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$.
Solution:

P	Q	R	Minterm	$P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$
T	T	T	$P \wedge Q \wedge R$	T	F	T	T
T	T	F	$P \wedge Q \wedge \neg R$	T	F	F	T
T	F	T	$P \wedge \neg Q \wedge R$	F	F	F	F
T	F	F	$P \wedge \neg Q \wedge \neg R$	F	F	F	F
F	T	T	$\neg P \wedge Q \wedge R$	F	T	T	T
F	T	F	$\neg P \wedge Q \wedge \neg R$	F	F	F	F
F	F	T	$\neg P \wedge \neg Q \wedge R$	F	T	F	T
F	F	F	$\neg P \wedge \neg Q \wedge \neg R$	F	F	F	F

The PDNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$ is

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R).$$

(b). Without constructing the truth table:

In order to obtain the principal disjunctive normal form of a given formula is constructed as follows:

(1). First replace →, by their equivalent formula containing only ∧, ∨ and ¬.

(2). Next, negations are applied to the variables by De Morgan's laws followed by the application of distributive laws.

(3). Any elementarily product which is a contradiction is dropped. Minterms are ob-tained in the disjunctions by introducing the missing factors. Identical minterms appearing in the disjunctions are deleted.

Example: Obtain the principal disjunctive normal form of

(a) $\neg P \vee Q$; (b) $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$.

Solution:

(a) $$\neg P \vee Q \Leftrightarrow (\neg P \wedge T) \vee (Q \wedge T) \qquad [\because A \wedge T \Leftrightarrow A]$$

$$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P)) \; [\because P \vee \neg P \Leftrightarrow T]$$

$$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$$

$$[\because P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$$

$$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q) \qquad [\because P \vee P \Leftrightarrow P]$$

(b) $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$$\Leftrightarrow (P \wedge Q \wedge T) \vee (\neg P \wedge R \wedge T) \vee (Q \wedge R \wedge T)$$

$$\Leftrightarrow (P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge R \wedge (Q \vee \neg Q)) \vee (Q \wedge R \wedge (P \vee \neg P))$$

$$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q)(\neg P \wedge R \wedge \neg Q)$$
$$\vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg P)$$

$$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$$

$$P \vee (P \wedge Q) \Leftrightarrow P$$

$$P \vee (\neg P \wedge Q) \Leftrightarrow P \vee Q$$

Solution: We write the principal disjunctive normal form of each formula and com-pare these normal forms.

(a) $P \vee (P \wedge Q) \Leftrightarrow (P \wedge T) \vee (P \wedge Q) \qquad [\because P \wedge Q \Leftrightarrow P]$

$$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (P \wedge Q) \qquad [\because P \vee \neg P \Leftrightarrow T]$$

$$\Leftrightarrow ((P \wedge Q) \vee (P \wedge \neg Q)) \vee (P \wedge Q) \text{ [by distributive laws]}$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \; [\because P \vee P \Leftrightarrow P]$$

which is the required PDNF.

Now, $$\Leftrightarrow P \wedge T$$

$$\Leftrightarrow P \wedge (Q \vee \neg Q)$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$$

which is the required PDNF.

Hence, $P \vee (P \wedge Q) \Leftrightarrow P$.

$(b)\ P \vee (\neg P \wedge Q) \Leftrightarrow (P \wedge T) \vee (\neg P \wedge Q)$

$$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (\neg P \wedge Q)$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

which is the required PDNF.

Now,

$$P \vee Q \Leftrightarrow (P \wedge T) \vee (Q \wedge T)$$

$$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

which is the required PDNF.

Hence, $\quad P \vee (\neg P \wedge Q) \Leftrightarrow P \vee Q.$

Example: Obtain the principal disjunctive normal form of

$$P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P)). \qquad \text{(Nov. 2011)}$$

Solution: Using $P \rightarrow Q \Leftrightarrow \neg P \vee Q$ and De Morgan's law, we obtain

$$\rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P)) \Leftrightarrow \neg P$$
$$\vee ((\neg P \vee Q) \wedge (Q \wedge P))$$

$$\Leftrightarrow \neg P \vee ((\neg P \wedge Q \wedge P) \vee (Q \wedge Q \wedge P)) \Leftrightarrow$$
$$\neg P \vee F \vee (P \wedge Q)$$

$$\Leftrightarrow \neg P \vee (P \wedge Q)$$

$$\Leftrightarrow (\neg P \wedge T) \vee (P \wedge Q)$$

$$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (P \wedge Q)$$

$$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$$

Hence $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$ is the required PDNF.

Principal Conjunctive Normal Form

The dual of a minterm is called a Maxterm. For a given number of variables, the *maxterm* consists of disjunctions in which each variable or its negation, but not both, appears only once. Each of the maxterm has the truth value F for exactly one com-bination of the truth values of the variables. Now we define the principal conjunctive normal form.

For a given formula, an equivalent formula consisting of conjunctions of the max-terms only is known as its *principle conjunctive normal form*. This normal form is also called the *product-of-sums canonical form.*The method for obtaining the PCNF for a given formula is similar to the one described previously for PDNF.

Example: Obtain the principal conjunctive normal form of the formula (¬P→R)∧(Q↔P)
Solution:

$$(\neg P \to R) \wedge (Q \leftrightarrow P)$$
$$\Leftrightarrow [\neg(\neg P) \vee R] \wedge [(Q \to P) \wedge (P \to Q)]$$
$$\Leftrightarrow (P \vee R) \wedge [(\neg Q \vee P) \wedge (\neg P \vee Q)]$$
$$\Leftrightarrow (P \vee R \vee F) \wedge [(\neg Q \vee P \vee F) \wedge (\neg P \vee Q \vee F)]$$
$$\Leftrightarrow [(P \vee R) \vee (Q \wedge \neg Q)] \wedge [\neg Q \vee P) \vee (R \wedge \neg R)] \wedge [(\neg P \vee Q) \vee (R \wedge \neg R)]$$
$$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R)$$
$$\wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$
$$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$

which is required principal conjunctive normal form.

Note: If the principal disjunctive (conjunctive) normal form of a given formula A containing n variables is known, then the principal disjunctive (conjunctive) normal form of $\neg A$ will consist of the disjunction (conjunction) of the remaining minterms (maxterms) which do not appear in the principal disjunctive (conjunctive) normal form of A. From $A \Leftrightarrow \neg\neg A$ one can obtain the principal conjunctive (disjunctive) normal form of A by repeated applications of De Morgan's laws to the principal disjunctive (conjunctive) normal form of $\neg A$.

Example: Find the PDNF form PCNF of $S : P \vee (\neg P \to (Q \vee (\neg Q \to R)))$.
Solution:

$$\Leftrightarrow P \vee (\neg P \to (Q \vee (\neg Q \to R)))$$
$$\Leftrightarrow P \vee (\neg(\neg P) \vee (Q \vee (\neg(\neg Q) \vee R))$$
$$\Leftrightarrow P \vee (P \vee Q \vee (Q \vee R)))$$
$$\Leftrightarrow P \vee (P \vee Q \vee R)$$
$$\Leftrightarrow P \vee Q \vee R$$

which is the PCNF.

Now PCNF of $\neg S$ is the conjunction of remaining maxterms, so

PCNF of $\neg S : (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R)$
$$\wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$$

Hence the PDNF of S is

$\neg$(PCNF of $\neg S$) : $(\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)$
$$\vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$$