Theory of Inference for Statement Calculus

Definition: The main aim of logic is to provide rules of inference to infer a conclusion from certain premises. The theory associated with rules of inference is known as inference theory .

Definition: If a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a deduction or a formal proof and the argument is called a *valid argument* or conclusion is called a *valid conclusion*.

Note: Premises means set of assumptions, axioms, hypothesis.

Definition: Let A and B be two statement formulas. We say that ”*B logically follows from A*” or ”B is a *valid conclusion (consequence)* of the premise A” iff $A \rightarrow B$ is a tautology, that is $A \Rightarrow B$. *We say that from a set of premises* $\{H_1, H_2, \cdots, H_m\}$, *a conclusion C follows logically iff*

$$H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow C$$

(1)

Note: To determine whether the conclusion logically follows from the given premises, we use the following methods:

- Truth table method
- Without constructing truth table method.

Validity Using Truth Tables

Given a set of premises and a conclusion, it is possible to determine whether the conclusion logically follows from the given premises by constructing truth tables as follows.

Let $P_1, P_2, \cdots, P_n$ be all the atomic variables appearing in the premises $H_1, H_2, \cdots, H_m$ and in the conclusion C. If all possible combinations of truth values are assigned to $P_1, P_2, \cdots, P_n$ and if the truth values of $H_1, H_2, ..., H_m$ and C are entered in a table. We look for the rows in which all $H_1, H_2, \cdots, H_m$ have the value T. If, for every such row, C also has the value T, then (1) holds. That is, the conclusion follows logically.

Alternatively, we look for the rows on which C has the value F. If, in every such row, at least one of the values of $H_1, H_2, \cdots, H_m$ is F, then (1) also holds. We call such a method a 'truth table technique' for the determination of the validity of a conclusion.

Example: Determine whether the conclusion C follows logically from the premises

H_1 and H_2.

(a) $H_1 : P \rightarrow Q$ $\quad H_2 : P \quad C : Q$

(b) $H_1 : P \rightarrow Q$ $\quad H_2 : \neg P \quad C : Q$

(c) $H_1 : P \rightarrow Q$ $\quad H_2 : \neg(P \wedge Q) \quad C : \neg P$

(d) $H_1 : \neg P$ $\quad H_2 : P \quad Q \quad C : \neg(P \wedge Q)$

(e) $H_1 : P \rightarrow Q$ $\quad H_2 : Q \quad C : P$

Solution: We first construct the appropriate truth table, as shown in table.

P	Q	$P \rightarrow Q$	$\neg P$	$\neg(P \wedge Q)$	$P \quad Q$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	T	F
F	F	T	T	T	T

(a) We observe that the first row is the only row in which both the premises have the value T. The conclusion also has the value T in that row. Hence it is valid.

In (b) the third and fourth rows, the conclusion Q is true only in the third row, but not in the fourth, and hence the conclusion is not valid.
Similarly, we can show that the conclusions are valid in (c) and (d) but not in (e).

Rules of Inference

The following are two important rules of inferences.

Rule P: A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is tautologically implied by one or more of the preceding formulas in the derivation.

Implication Formulas

$I_1 : P \wedge Q \Rightarrow P$ (simplification)

$I_2 : P \wedge Q \Rightarrow Q$

$I_3 : P \Rightarrow P \vee Q$

$I_4 : Q \Rightarrow P \vee Q$

$I_5 : \neg P \Rightarrow P \rightarrow Q$

$I_6 : Q \Rightarrow P \rightarrow Q$

$I_7 : \neg(P \rightarrow Q) \Rightarrow P$

$I_8 : \neg(P \rightarrow Q) \Rightarrow \neg Q$

$I_9 : P, Q \Rightarrow P \wedge Q$

$I_{10} : \neg P, P \vee Q \Rightarrow Q$ (disjunctive syllogism)

$I_{11} : P, P \rightarrow Q \Rightarrow Q$ (modus ponens)

$I_{12} : \neg Q, P \rightarrow Q \Rightarrow \neg P$ (modus tollens)

$I_{13} : P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$ (hypothetical syllogism)

$I_{14} : P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$ (dilemma)

Example: Demonstrate that R is a valid inference from the premises $P \rightarrow Q$, $Q \rightarrow R$, and P.
Solution:

$\{1\}$	(1)	$P \rightarrow Q$	Rule P
$\{2\}$	(2)	P	Rule P,
$\{1, 2\}$	(3)	Q	Rule T, (1), (2), and I_{13}
$\{4\}$	(4)	$Q \rightarrow R$	Rule P
$\{1, 2, 4\}$	(5)	R	Rule T, (3), (4), and I_{13}

Hence the result.

Example: Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $\neg H \rightarrow (A \wedge \neg B)$, and $(A \wedge \neg B) \rightarrow (R \vee S)$.

Solution:

$\{1\}$	(1)	$(C \vee D) \rightarrow \neg H$	Rule P
$\{2\}$	(2)	$\neg H \rightarrow (A \wedge \neg B)$	Rule P
$\{1, 2\}$	(3)	$(C \vee D) \rightarrow (A \wedge \neg B)$	Rule T, (1), (2), and I_{13}
$\{4\}$	(4)	$(A \wedge \neg B) \rightarrow (R \vee S)$	Rule P
$\{1, 2, 4\}$	(5)	$(C \vee D) \rightarrow (R \vee S)$	Rule T, (3), (4), and I_{13}
$\{6\}$	(6)	$C \vee D$	Rule P
$\{1, 2, 4, 6\}$	(7)	$R \vee S$	Rule T, (5), (6), and I_{11}

Hence the result.

Example: Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$.

Solution:

$\{1\}$	(1)	$P \vee Q$	Rule P
$\{1\}$	(2)	$\neg P \rightarrow Q$	Rule T, (1) $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
$\{3\}$	(3)	$Q \rightarrow S$	Rule P
$\{1, 3\}$	(4)	$\neg P \rightarrow S$	Rule T, (2), (3), and I_{13}
$\{1, 3\}$	(5)	$\neg S \rightarrow P$	Rule T, (4), $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
$\{6\}$	(6)	$P \rightarrow R$	Rule P
$\{1, 3, 6\}$	(7)	$\neg S \rightarrow R$	Rule T, (5), (6), and I_{13}
$\{1, 3, 6\}$	(8)	$S \vee R$	Rule T, (7) and $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

Hence the result.

Example: Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$,

$Q \rightarrow R$, $P \rightarrow M$, and $\neg M$.

Solution:

$\{1\}$	(1)	$P \rightarrow M$	Rule P
$\{2\}$	(2)	$\neg M$	Rule P
$\{1, 2\}$	(3)	$\neg P$	Rule T, (1), (2), and I_{12}
$\{4\}$	(4)	$P \vee Q$	Rule P
$\{1, 2, 4\}$	(5)	Q	Rule T, (3), (4), and I_{10}
$\{6\}$	(6)	$Q \rightarrow R$	Rule P

$\{1, 2, 4, 6\}$	(7)	R	Rule T, (5), (6), and I_{11}
$\{1, 2, 4, 6\}$	(8)	$R \wedge (P \vee Q)$	Rule T, (4), (7) and I_9

Hence the result.

Example: Show I_{12} : $\neg Q, P \to Q \Rightarrow \neg P$.
Solution:

$\{1\}$	(1)	$P \to Q$	Rule P
$\{1\}$	(2)	$\neg Q \to \neg P$	Rule T, (1), and $P \to Q \Leftrightarrow \neg Q \to \neg P$
$\{3\}$	(3)	$\neg Q$	Rule P
$\{1, 3\}$	(4)	$\neg P$	Rule T, (2), (3), and I_{11}

Hence the result.

Example: Test the validity of the following argument:

"If you work hard, you will pass the exam. You did not pass. Therefore, you did not work hard".

Example: Test the validity of the following statements:

"If Sachin hits a century, then he gets a free car. Sachin does not get a free car.

Therefore, Sachin has not hit a century".

Rules of Conditional Proof or Deduction Theorem

We shall now introduce a third inference rule, known as CP or rule of conditional proof.
Rule CP: If we can derive S from R and a set of premises, then we can derive $R \to S$ from the set of premises alone.
Rule CP is not new for our purpose her because it follows from the equivalence

$$(P \wedge R) \to S \Leftrightarrow P \to (R \to S)$$

Let P denote the conjunction of the set of premises and let R be any formula. The above equivalence states that if R is included as an additional premise and S is derived from $P \wedge R$, then $R \to S$ can be derived from the premises P alone.

Rule CP is also called the *deduction theorem* and is generally used if the conclu-sion of the form $R \to S$. In such cases, R is taken as an additional premise and S is derived from the given premises and R.

Example: Show that $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\neg R \vee P$, and Q.
(Nov. 2011)

Solution: Instead of deriving $R \to S$, we shall include R as an additional premise and show S first.

{1}	(1) $\neg R \vee P$	Rule P
{2}	(2) R	Rule P (assumed premise)
{1, 2}	(3) P	Rule T, (1), (2), and I_{10}
{4}	(4) $P \to (Q \to S)$	Rule P
{1, 2, 4}	(5) $Q \to S$	Rule T, (3), (4), and I_{11}
{6}	(6) Q	Rule P
{1, 2, 4, 6}	(7) S	Rule T, (5), (6), and I_{11}
{1, 2, 4, 6}	(8) $R \to S$	Rule CP

Example: Show that $P \to S$ can be derived from the premises $\neg P \vee Q$, $\neg Q \vee R$, and $R \to S$.

Solution: We include P as an additional premise and derive S.

{1}	(1) $\neg P \vee Q$	Rule P
{2}	(2) P	Rule P (assumed premise)
{1, 2}	(3) Q	Rule T, (1), (2), and I_{10}
{4}	(4) $\neg Q \vee R$	Rule P
{1, 2, 4}	(5) R	Rule T, (3), (4), and I_{10}
{6}	(6) $R \to S$	Rule P
{1, 2, 4, 6}	(7) S	Rule T, (5), (6), and I_{11}
{1, 2, 4, 6}	(8) $P \to S$	Rule CP

Example: 'If there was a ball game, then traveling was difficult. If they arrived on time, then traveling was not difficult. They arrived on time. Therefore, there was no ball game'. Show that these statements constitute a valid argument. Solution: Let us indicate the statements as follows:

P : There was a ball game.
Q: Traveling was difficult.
R: They arrived on time.

Hence, the given premises are $P \to Q$, $R \to \neg Q$, and R. The conclusion is $\neg P$.

$\{1\}$	(1) $R \to \neg Q$	Rule P
$\{2\}$	(2) R	Rule P
$\{1, 2\}$	(3) $\neg Q$	Rule T, (1), (2), and I_{11}
$\{4\}$	(4) $P \to Q$	Rule P
$\{4\}$	(5) $\neg Q \to \neg P$	Rule T, (4), and $P \to Q \Leftrightarrow \neg Q \to \neg P$
$\{1, 2, 4\}$	(6) $\neg P$	Rule T, (3), (5), and I_{11}

Example: By using the method of derivation, show that following statements con-stitute a valid argument: ”If A works hard, then either B or C will enjoy. If B enjoys, then A will not work hard. If D enjoys, then C will not. Therefore, if A works hard, D will not enjoy.

Solution: Let us indicate statements as follows:

Given premises are $P \to (Q \vee R)$, $Q \to \neg P$, and $S \to \neg R$. The conclusion is $P \to \neg S$.
We include P as an additional premise and derive $\neg S$.

$\{1\}$	(1) P	Rule P (additional premise)
$\{2\}$	(2) $P \to (Q \vee R)$	Rule P
$\{1, 2\}$	(3) $Q \vee R$	Rule T, (1), (2), and I_{11}
$\{1, 2\}$	(4) $\neg Q \to R$	Rule T, (3) and $P \to Q \Leftrightarrow P \vee Q$
$\{1, 2\}$	(5) $\neg R \to Q$	Rule T, (4), and $P \to Q \Leftrightarrow \neg Q \to \neg P$
$\{6\}$	(6) $Q \to \neg P$	Rule P
$\{1, 2, 6\}$	(7) $\neg R \to \neg P$	Rule T, (5), (6), and I_{13}
$\{1, 2, 6\}$	(8) $P \to R$	Rule T, (7) and $P \to Q \Leftrightarrow \neg Q \to \neg P$
$\{9\}$	(9) $S \to \neg R$	Rule P
$\{9\}$	(10) $R \to \neg S$	Rule T, (9) and $P \to Q \Leftrightarrow \neg Q \to \neg P$
$\{1, 2, 6, 9\}$	(11) $P \to \neg S$	Rule T, (8), (10) and I_{13}
$\{1, 2, 6, 9\}$	(12) $\neg S$	Rule T, (1), (11) and I_{11}

Example: Determine the validity of the following arguments using propositional logic:
”Smoking is healthy. If smoking is healthy, then cigarettes are prescribed by physicians. Therefore, cigarettes are prescribed by physicians”. (May-2012)

Solution: Let us indicate the statements as follows:

P : Smoking is healthy.

Q: Cigarettes are prescribed by physicians.

Hence, the given premises are P , $P \to Q$. The conclusion is Q.

$\{1\}$	(1) $P \to Q$	Rule P
$\{2\}$	(2) P	Rule P

{1, 2} (3) Q Rule T, (1), (2), and I_{11}

Hence, the given statements constitute a valid argument.

Consistency of Premises

A set of formulas $H_1, H_2, \cdots, H_m$ is said to be *consistent* if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in $H_1, H_2, \cdots, H_m$.

If, for every assignment of the truth values to the atomic variables, at least one of the formulas $H_1, H_2, \cdots, H_m$ is false, so that their conjunction is identically false, then the formulas $H_1, H_2, \cdots, H_m$ are called *inconsistent*.

Alternatively, a set of formulas $H_1, H_2, \cdots, H_m$ is inconsistent if their conjunction implies a contradiction, that is,

$$H_1 \wedge H_2 \wedge \cdots \wedge H_m \Rightarrow R \wedge \neg R$$

where R is any formula.

Example: Show that the following premises are inconsistent:

(1). If Jack misses many classes through illness, then he fails high school.
(2). If Jack fails high school, then he is uneducated.
(3). If Jack reads a lot of books, then he is not uneducated.
(4). Jack misses many classes through illness and reads a lot of books.

Solution: Let us indicate the statements as follows:

E: Jack misses many classes through illness.
S: Jack fails high school.
A: Jack reads a lot of books.
H: Jack is uneducated.

The premises are $E \to S$, $S \to H$, $A \to \neg H$, and $E \wedge A$.

{1}	(1) $E \to S$	Rule P
{2}	(2) $S \to H$	Rule P
{1, 2}	(3) $E \to H$	Rule T, (1), (2), and I_{13}
{4}	(4) $A \to \neg H$	Rule P
{4}	(5) $H \to \neg A$	Rule T, (4), and $P \to Q \Leftrightarrow \neg Q \to \neg P$
{1, 2, 4}	(6) $E \to \neg A$	Rule T, (3), (5), and I_{13}
{1, 2, 4}	(7) $\neg E \vee \neg A$	Rule T, (6) and $P \to Q \Leftrightarrow \neg P \vee Q$
{1, 2, 4}	(8) $\neg(E \wedge A)$	Rule T, (7), and $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
{9}	(9) $E \wedge A$	Rule P
{1, 2, 4, 9}	(10) $\neg(E \wedge A) \wedge (E \wedge A)$	Rule T, (8), (9) and I_9

Thus, the given set of premises leads to a contradiction and hence it is inconsistent.

Example: Show that the following set of premises is inconsistent: ”If the contract is valid, then John is liable for penalty. If John is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact, the contract is valid, and the bank will loan him money.”

Solution: Let us indicate the statements as follows:

V : The contract is valid.

L: John is liable for penalty.

M: Bank will loan him money.

B: John will go bankrupt.

$\{1\}$	(1)	$V \to L$	Rule P
$\{2\}$	(2)	$L \to B$	Rule P
$\{1, 2\}$	(3)	$V \to B$	Rule T, (1), (2), and I_{13}
$\{4\}$	(4)	$M \to \neg B$	Rule P
$\{4\}$	(5)	$M \to \neg M$	Rule T, (4), and $P \to Q \Leftrightarrow \neg Q \to \neg P$
$\{1, 2, 4\}$	(6)	$V \to \neg M$	Rule T, (3), (5), and I_{13}
$\{1, 2, 4\}$	(7)	$\neg V \vee \neg M$	Rule T, (6) and $P \to Q \Leftrightarrow \neg P \vee Q$
$\{1, 2, 4\}$	(8)	$\neg(V \wedge M)$	Rule T, (7), and $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
$\{9\}$	(9)	$V \wedge M$	Rule P
$\{1, 2, 4, 9\}$	(10)	$\neg(V \wedge M) \wedge (V \wedge M)$	Rule T, (8), (9) and I_9

Thus, the given set of premises leads to a contradiction and hence it is inconsistent.

Indirect Method of Proof

The method of using the rule of conditional proof and the notion of an inconsistent set of premises is called the *indirect method of proof* or *proof by contradiction*.

In order to show that a conclusion C follows logically from the premises $H_1, H_2, \cdots,$ H_m, we assume that C is false and consider $\neg C$ as an additional premise. If the new set of premises is inconsistent, so that they imply a contradiction. Therefore, the assump-tion that $\neg C$ is true does not hold.

Hence, C is true whenever $H_1, H_2, \cdots, H_m$ are true. Thus, C follows logically from the premises $H_1, H_2, \cdots, H_m$.

Example: Show that $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.

Solution: We introduce $\neg\neg(P \wedge Q)$ as additional premise and show that this additional premise leads to a contradiction.

{1}	(1) $\neg\neg(P \wedge Q)$	Rule P (assumed)
{1}	(2) $P \wedge Q$	Rule T, (1), and $\neg\neg P \Leftrightarrow P$
{1}	(3) P	Rule T, (2), and I_1
{4}	(4) $\neg P \wedge \neg Q$	Rule P
{4}	(5) $\neg P$	Rule T, (4), and I_1
{1, 4}	(6) $P \wedge \neg P$	Rule T, (3), (5), and I_9

Hence, our assumption is wrong.

Thus, $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.

Example: Using the indirect method of proof, show that

$$P \rightarrow Q,\ Q \rightarrow R,\ \neg(P \wedge R),\ P \vee R \Rightarrow R.$$

Solution: We include $\neg R$ as an additional premise. Then we show that this leads to a contradiction.

{1}	(1) $P \rightarrow Q$	Rule P
{2}	(2) $Q \rightarrow R$	Rule P
{1, 2}	(3) $P \rightarrow R$	Rule T, (1), (2), and I_{13}
{4}	(4) $\neg R$	Rule P (assumed)
{1, 2, 4}	(5) $\neg P$	Rule T, (4), and I_{12}
{6}	(6) $P \vee R$	Rule P
{1, 2, 4, 6}	(7) R	Rule T, (5), (6) and I_{10}
{1, 2, 4, 6}	(8) $R \wedge \neg R$	Rule T, (4), (7), and I_9

Hence, our assumption is wrong.

Example: Show that the following set of premises are inconsistent, using proof by contradiction

$$P \rightarrow (Q \vee R),\ Q \rightarrow \neg P,\ S \rightarrow \neg R,\ P \Rightarrow P \rightarrow \neg S.$$

Solution: We include $\neg(P \rightarrow \neg S)$ as an additional premise. Then we show that this leads to a contradiction.

$$\therefore \neg(P \rightarrow \neg S) \Leftrightarrow \neg(\neg P \vee \neg S) \Leftrightarrow P \wedge S.$$

{1}	(1) $P \rightarrow (Q \vee R)$	Rule P
{2}	(2) P	Rule P
{1, 2}	(3) $Q \vee R$	Rule T, (1), (2), and Modus Ponens
{4}	(4) $P \wedge S$	Rule P (assumed)
{1, 2, 4}	(5) S	Rule T, (4), and $P \wedge Q \Rightarrow P$

{6}	(6) $S \to \neg R$	Rule P
{1, 2, 4, 6}	(7) $\neg R$	Rule T, (5), (6) and Modus Ponens
{1, 2, 4, 6}	(8) Q	Rule T, (3), (7), and $P \wedge Q, \neg Q \Rightarrow P$
{9}	(9) $Q \to \neg P$	Rule P
{1, 2, 4, 6}	(10) $\neg P$	Rule T, (8), (9), and $P \wedge Q, \neg Q \Rightarrow P$
{1, 2, 4, 6}	(11) $P \wedge \neg P$	Rule T, (2), (10), and $P, Q \Rightarrow P \wedge Q$
{1, 2, 4, 6}	(12) F	Rule T, (11), and $P \wedge \neg P \Leftrightarrow F$

Hence, it is proved that the given premises are inconsistent.

The Predicate Calculus

Predicate

A part of a declarative sentence describing the properties of an object is called a predicate. The logic based upon the analysis of predicate in any statement is called predicate logic.

Consider two statements:

John is a bachelor
Smith is a bachelor.

In each statement "is a bachelor" is a predicate. Both John and Smith have the same property of being a bachelor. In the statement logic, we require two different symbols to express them and these symbols do not reveal the common property of these statements. In predicate calculus these statements can be replaced by a single statement "x is a bachelor". A predicate is symbolized by a capital letters which is followed by the list of variables. The list of variables is enclosed in parenthesis. If P stands for the predicate "is a bachelor", then $P(x)$ stands for "x is a bachelor",where x is a predicate variable.

`The domain for $P(x)$: x is a bachelor, can be taken as the set of all human names. Note that $P(x)$ is not a statement, but just an expression. Once a value is assigned to x, $P(x)$ becomes a statement and has the truth value. If x is Ram, then $P(x)$ is a statement and its truth value is true.

Quantifiers

Quantifiers: Quantifiers are words that are refer to quantities such as 'some' or 'all'.

Universal Quantifier: The phrase 'forall' (denoted by $\forall$) is called the universal quantifier.

For example, consider the sentence "All human beings are mortal".

Let $P(x)$ denote 'x is a mortal'.

Then, the above sentence can be written as

$$(\forall x \in S)P(x) \text{ or } \forall x P(x)$$

where S denote the set of all human beings.

$\forall x$ represents each of the following phrases, since they have essentially the same for all x

For every x
For each x.

Existential Quantifier: The phrase 'there exists' (denoted by $\exists$) is called the exis-tential quantifier.

For example, consider the sentence

"There exists x such that $x^2 = 5$.

This sentence can be written as

$$(\exists x \in R)P(x) \text{ or } (\exists x)P(x),$$

where $P(x) : x^2 = 5$.

$\exists x$ represents each of the following phrases

There exists an x
There is an x
For some x
There is at least one x.

Example: Write the following statements in symbolic form:

(i). Something is good
(ii). Everything is good
(iii). Nothing is good
(iv). Something is not good.

Solution: Statement (i) means "There is atleast one x such that, x is good".
Statement (ii) means "Forall x, x is good".
Statement (iii) means, "Forall x, x is not good".
Statement (iv) means, "There is atleast one x such that, x is not good.
Thus, if $G(x)$: x is good, then

statement (i) can be denoted by $(\exists x)G(x)$

statement (ii) can be denoted by $(\forall x)G(x)$

statement (iii) can be denoted by $(\forall x)\neg G(x)$

statement (iv) can be denoted by $(\exists x)\neg G(x)$.

Example: Let $K(x)$: x is a man
$L(x)$: x is mortal
$M(x)$: x is an integer
$N(x)$: x either positive or negative

Express the following using quantifiers:

- All men are mortal
- Any integer is either positive or negative.

Solution: (a) The given statement can be written as
for all x, if x is a man, then x is mortal and this can be expressed as
$(x)(K(x) \to L(x))$.

(b) The given statement can be written as
for all x, if x is an integer, then x is either positive or negative and this can be expressed as $(x)(M(x) \to N(x))$.

Free and Bound Variables

Given a formula containing a part of the form $(x)P(x)$ or $(\exists x)P(x)$, such a part is called an x-bound part of the formula. Any occurrence of x in an x-bound part of the formula is called a bound occurrence of x, while any occurrence of x or of any variable that is not a bound occurrence is called a free occurrence. The smallest formula immediately following $(\forall x)$ or $(\exists x)$ is called the scope of the quantifier.

Consider the following formulas:

- $(x)P(x, y)$
- $(x)(P(x) \rightarrow Q(x))$
- $(x)(P(x) \rightarrow (\exists y)R(x, y))$
- $(x)(P(x) \rightarrow R(x)) \vee (x)(R(x) \rightarrow Q(x))$
- $(\exists x)(P(x) \wedge Q(x))$
- $(\exists x)P(x) \wedge Q(x)$.

In (1), $P(x, y)$ is the scope of the quantifier, and occurrence of x is bound occurrence, while the occurrence of y is free occurrence.

In (2), the scope of the universal quantifier is $P(x) \rightarrow Q(x)$, and all concrescences of x are bound.

In (3), the scope of (x) is $P(x) \rightarrow (\exists y)R(x, y)$, while the scope of $(\exists y)$ is $R(x, y)$. All occurrences of both x and y are bound occurrences.

In (4), the scope of the first quantifier is $P(x) \rightarrow R(x)$ and the scope of the second is $R(x) \rightarrow Q(x)$. All occurrences of x are bound occurrences.

In (5), the scope $(\exists x)$ is $P(x) \wedge Q(x)$.

In (6), the scope of $(\exists x)$ is $P(x)$ and the last of occurrence of x in $Q(x)$ is free.

Negations of Quantified Statements

(i). $\neg(x)P(x) \Leftrightarrow (\exists x)\neg P(x)$

(ii). $\neg(\exists x)P(x) \Leftrightarrow (x)(\neg P(x))$.

Example: Let $P(x)$ denote the statement ”x is a professional athlete” and let $Q(x)$ denote the statement ”x plays soccer”. The domain is the set of all people.

(a). Write each of the following proposition in English.

- $(x)(P(x) \rightarrow Q(x)$
- $(\exists x)(P(x) \wedge Q(x))$
- $(x)(P(x) \vee Q(x))$

(b). Write the negation of each of the above propositions, both in symbols and in words.

Solution:

(a). (i). For all x, if x is an professional athlete then x plays soccer.

”All professional athletes plays soccer” or ”Every professional athlete plays soccer”.

(ii). There exists an x such that x is a professional athlete and x plays soccer.

"Some professional athletes paly soccer".

(iii). For all x, x is a professional athlete or x plays soccer.

"Every person is either professional athlete or plays soccer".

(b). (i). In symbol: We know that

$$\neg(x)(P(x) \to Q(x)) \Leftrightarrow (\exists x)\neg(P(x) \to Q(x)) \Leftrightarrow (\exists x)\neg(\neg(P(x)) \vee Q(x))$$

$$\Leftrightarrow (\exists x)(P(x) \wedge \neg Q(x))$$

There exists an x such that, x is a professional athlete and x does not paly soccer.
In words: "Some professional athlete do not play soccer".

(ii). $\neg(\exists x)(P(x) \wedge Q(x)) \Leftrightarrow (x)(\neg P(x) \vee \neg Q(x))$

In words: "Every people is neither a professional athlete nor plays soccer" or All people either not a professional athlete or do not play soccer".

(iii). $\neg(x)(P(x) \vee Q(x)) \Leftrightarrow (\exists x)(\neg P(x) \wedge \neg Q(x))$.

In words: "Some people are not professional athlete or do not paly soccer".

Inference Theory of the Predicate Calculus

To understand the inference theory of predicate calculus, it is important to be famil-iar with the following rules:

Rule US: Universal specification or instaniation

$$(x)A(x) \Rightarrow A(y)$$

From $(x)A(x)$, one can conclude $A(y)$.

Rule ES: Existential specification

$$(\exists x)A(x) \Rightarrow A(y)$$

From $(\exists x)A(x)$, one can conclude $A(y)$.

Rule EG: Existential generalization

$$A(x) \Rightarrow (\exists y)A(y)$$

From $A(x)$, one can conclude $(\exists y)A(y)$.

Rule UG: Universal generalization

$$A(x) \Rightarrow (y)A(y)$$

From $A(x)$, one can conclude $(y)A(y)$.

Equivalence formulas:

$E_{31} : (\exists x)[A(x) \vee B(x)] \Leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$

$E_{32} : (x)[A(x) \wedge B(x)] \Leftrightarrow (x)A(x) \wedge (x)B(x)$

$E_{33} : \neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)$

$E_{34} : \neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$

$E_{35} : (x)(A \vee B(x)) \Leftrightarrow A \vee (x)B(x)$

$E_{36} : (\exists x)(A \wedge B(x)) \Leftrightarrow A \wedge (\exists x)B(x)$

$E_{37} : (x)A(x) \to B \Leftrightarrow (x)(A(x) \to B)$

$E_{38} : (\exists x)A(x) \to B \Leftrightarrow (x)(A(x) \to B)$

$E_{39} : A \to (x)B(x) \Leftrightarrow (x)(A \to B(x))$

$E_{40} : A \rightarrow (\exists x)B(x) \Leftrightarrow (\exists x)(A \rightarrow B(x))$

$E_{41} : (\exists x)(A(x) \rightarrow B(x)) \Leftrightarrow (x)A(x) \rightarrow (\exists x)B(x)$

$E_{42} : (\exists x)A(x) \rightarrow (x)B(X) \Leftrightarrow (x)(A(x) \rightarrow B(X))$.

Example: Verify the validity of the following arguments:
"All men are mortal. Socrates is a man. Therefore, Socrates is mortal".

or

Show that $(x)[H(x) \rightarrow M(x)] \wedge H(s) \Rightarrow M(s)$.

Solution: Let us represent the statements as follows:
$H(x)$: x is a man
$M(x)$: x is a mortal
s : Socrates

Thus, we have to show that $(x)[H(x) \rightarrow M(x)] \wedge H(s) \Rightarrow M(s)$.

{1}	(1)	$(x)[H(x) \rightarrow M(x)]$	Rule P
{1}	(2)	$H(s) \rightarrow M(s)$	Rule US, (1)
{3}	(3)	$H(s)$	Rule P
{1, 3}	(4)	$M(s)$	Rule T, (2), (3), and I_{11}

Example: Establish the validity of the following argument:"All integers are ratio-nal numbers. Some integers are powers of 2. Therefore, some rational numbers are powers of 2".

Solution: Let $P(x)$: x is an integer

$R(x)$: x is rational number

$S(x)$: x is a power of 2

Hence, the given statements becomes

$$(x)(P(x) \rightarrow R(x)), (\exists x)(P(x) \wedge S(x)) \Rightarrow (\exists x)(R(x) \wedge S(x))$$

Solution:

{1}	(1)	$(\exists x)(P(x) \wedge S(x))$	Rule P
{1}	(2)	$P(y) \wedge S(y)$	Rule ES, (1)
{1}	(3)	$P(y)$	Rule T, (2) and $P \wedge Q \Rightarrow P$
{1}	(4)	$S(y)$	Rule T, (2) and $P \wedge Q \Rightarrow Q$
{5}	(5)	$(x)(P(x) \rightarrow R(x))$	Rule P
{5}	(6)	$P(y) \rightarrow R(y)$	Rule US, (5)
{1, 5}	(7)	$R(y)$	Rule T, (3), (6) and $P, P \rightarrow Q \Rightarrow Q$
{1, 5}	(8)	$R(y) \wedge S(y)$	Rule T, (4), (7) and $P, Q \Rightarrow P \wedge Q$
{1, 5}	(9)	$(\exists x)(R(x) \wedge S(x))$	Rule EG, (8)

Hence, the given statement is valid.

Example: Show that $(x)(P(x) \rightarrow Q(x)) \wedge (x)(Q(x) \rightarrow R(x)) \Rightarrow (x)(P(x) \rightarrow R(x))$.

Solution:

$\{1\}$	(1) $(x)(P(x) \rightarrow Q(x))$	Rule P
$\{1\}$	(2) $P(y) \rightarrow Q(y)$	Rule US, (1)
$\{3\}$	(3) $(x)(Q(x) \rightarrow R(x))$	Rule P
$\{3\}$	(4) $Q(y) \rightarrow R(y)$	Rule US, (3)
$\{1, 3\}$	(5) $P(y) \rightarrow R(y)$	Rule T, (2), (4), and I_{13}
$\{1, 3\}$	(6) $(x)(P(x) \rightarrow R(x))$	Rule UG, (5)

Example: Show that $(\exists x)M(x)$ follows logically from the premises

$(x)(H(x) \rightarrow M(x))$ and $(\exists x)H(x)$.

Solution:

$\{1\}$	(1) $(\exists x)H(x)$	Rule P
$\{1\}$	(2) $H(y)$	Rule ES, (1)
$\{3\}$	(3) $(x)(H(x) \rightarrow M(x))$	Rule P
$\{3\}$	(4) $H(y) \rightarrow M(y)$	Rule US, (3)
$\{1, 3\}$	(5) $M(y)$	Rule T, (2), (4), and I_{11}
$\{1, 3\}$	(6) $(\exists x)M(x)$	Rule EG, (5)

Hence, the result.

Example: Show that $(\exists x)[P(x) \wedge Q(x)] \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$.

Solution:

$\{1\}$	(1) $(\exists x)(P(x) \wedge Q(x))$	Rule P
$\{1\}$	(2) $P(y) \wedge Q(y)$	Rule ES, (1)
$\{1\}$	(3) $P(y)$	Rule T, (2), and I_1
$\{1\}$	(4) $(\exists x)P(x)$	Rule EG, (3)
$\{1\}$	(5) $Q(y)$	Rule T, (2), and I_2
$\{1\}$	(6) $(\exists x)Q(x)$	Rule EG, (5)
$\{1\}$	(7) $(\exists x)P(x) \wedge (\exists x)Q(x)$	Rule T, (4), (5) and I_9

Hence, the result.

Note: Is the converse true?

$\{1\}$	(1) $(\exists x)P(x) \wedge (\exists x)Q(x)$	Rule P
$\{1\}$	(2) $(\exists x)P(x)$	Rule T, (1) and I_1
$\{1\}$	(3) $(\exists x)Q(x)$	Rule T, (1), and I_1
$\{1\}$	(4) $P(y)$	Rule ES, (2)
$\{1\}$	(5) $Q(s)$	Rule ES, (3)

Here in step (4), y is fixed, and it is not possible to use that variable again in step (5). Hence, the *converse is not true.*

Example: Show that from $(\exists x)[F(x) \wedge S(x)] \rightarrow (y)[M(y) \rightarrow W(y)]$ and $(\exists y)[M(y) \wedge \neg W(y)]$ the conclusion $(x)[F(x) \rightarrow \neg S(x)]$ follows.

{1}	(1)	$(\exists y)[M(y) \wedge \neg W(y)]$	Rule P
{1}	(2)	$[M(z) \wedge \neg W(z)]$	Rule ES, (1)
{1}	(3)	$\neg[M(z) \rightarrow W(z)]$	Rule T, (2), and $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$
{1}	(4)	$(\exists y)\neg[M(y) \rightarrow W(y)]$	Rule EG, (3)
{1}	(5)	$\neg(y)[M(y) \rightarrow W(y)]$	Rule T, (4), and $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
{1}	(6)	$(\exists x)[F(x) \wedge S(x)] \rightarrow (y)[M(y) \rightarrow W(y)]$	Rule P
{1, 6}	(7)	$\neg(\exists x)[F(x) \wedge S(x)]$	Rule T, (5), (6) and I_{12}
{1, 6}	(8)	$(x)\neg[F(x) \wedge S(x)]$	Rule T, (7), and $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
{1, 6}	(9)	$\neg[F(z) \wedge S(z)]$	Rule US, (8)
{1, 6}	(10)	$\neg F(z) \vee \neg S(z)$	Rule T, (9), and De Morgan's laws
{1, 6}	(11)	$F(z) \rightarrow \neg S(z)$	Rule T, (10), and $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
{1, 6}	(12)	$(x)(F(x) \rightarrow \neg S(x))$	Rule UG, (11)

Hence, the result.

Example: Show that $(x)(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$. (May. 2012)

Solution: We shall use the indirect method of proof by assuming $\neg((x)P(x) \vee (\exists x)Q(x))$ as an additional premise.

{1}	(1)	$\neg((x)P(x) \vee (\exists x)Q(x))$	Rule P (assumed)
{1}	(2)	$\neg(x)P(x) \wedge \neg(\exists x)Q(x)$	Rule T, (1) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
{1}	(3)	$\neg(x)P(x)$	Rule T, (2), and I_1
{1}	(4)	$(\exists x)\neg P(x)$	Rule T, (3), and $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
{1}	(5)	$\neg(\exists x)Q(x)$	Rule T, (2), and I_2
{1}	(6)	$(x)\neg Q(x)$	Rule T, (5), and $\neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)$
{1}	(7)	$\neg P(y)$	Rule ES, (5), (6) and I_{12}
{1}	(8)	$\neg Q(y)$	Rule US, (6)
{1}	(9)	$\neg P(y) \wedge \neg Q(y)$	Rule T, (7), (8)and I_9
{1}	(10)	$\neg(P(y) \vee Q(y))$	Rule T, (9), and $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
{11}	(11)	$(x)(P(x) \vee Q(x))$	Rule P
{11}	(12)	$(P(y) \vee Q(y))$	Rule US
{1, 11}	(13)	$\neg(P(y) \vee Q(y)) \wedge (P(y) \vee Q(y))$	Rule T, (10), (11), and I_9
{1, 11}	(14)	F	Rule T, and (13)

which is a contradiction.Hence, the statement is valid.

Example: Using predicate logic, prove the validity of the following argument: "Every husband argues with his wife. x is a husband. Therefore, x argues with his wife".

Solution: Let $P(x)$: x is a husband.

$Q(x)$: x argues with his wife.

Thus, we have to show that $(x)[P(x) \rightarrow Q(x)] \wedge P(x) \Rightarrow Q(y)$.

{1}	(1)	$(x)(P(x) \rightarrow Q(x))$	Rule P
{1}	(2)	$P(y) \rightarrow Q(y)$	Rule US, (1)
{1}	(3)	$P(y)$	Rule P
{1}	(4)	$Q(y)$	Rule T, (2), (3), and I_{11}

Example: Prove using rules of inference

Duke is a Labrador retriever.
All Labrador retriever like to swim.
Therefore Duke likes to swim.

Solution: We denote

$L(x)$: x is a Labrador retriever.

$S(x)$: x likes to swim.

d: Duke.

We need to show that $L(d) \wedge (x)(L(x) \rightarrow S(x)) \Rightarrow S(d)$.

{1}	(1)	$(x)(L(x) \rightarrow S(x))$	Rule P
{1}	(2)	$L(d) \rightarrow S(d)$	Rule US, (1)
{2}	(3)	$L(d)$	Rule P
{1, 2}	(4)	$S(d)$	Rule T, (2), (3), and I_{11}.

JNTUK Previous questions

1. Test the Validity of the Following argument: "All dogs are barking. Some animals are dogs. Therefore, some animals are barking".
2. Test the Validity of the Following argument:
   "Some cats are animals. Some dogs are animals. Therefore, some cats are dogs".
3. Symbolizes and prove the validity of the following arguments :
   (i) Himalaya is large. Therefore every thing is large.
   (ii) Not every thing is edible. Therefore nothing is edible.
4. a) Find the PCNF of (~p↔r) ^(q↔p) ?
   b) Explain in brief about duality Law?

   c) Construct the Truth table for ~(~p^~q)?
   d) Find the disjunctive Normal form of ~(p → (q^r)) ?
5. Define Well Formed Formula? Explain about Tautology with example?
6. Explain in detail about the Logical Connectives with Examples?

7. Obtain the principal conjunctive normal form of the formula $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$
8. Prove that $(\exists x)P(x) \wedge Q(x) \rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$. Does the converse hold?
9. Show that from i) $(\exists x)(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$
   ii) $(\exists y)\ (M(y) \wedge \neg W(y))$ the conclusion $(x)(F(x) \rightarrow \neg S(x))$ follows.

10. Obtain the principal disjunctive and conjunctive normal forms of $(P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg Q \wedge \neg R))$. Is this formula a tautology?
11. Prove that the following argument is valid: No Mathematicians are fools. No one who is not a fool is an administrator. Sitha is a mathematician. Therefore Sitha is not an administrator.
12. Test the Validity of the Following argument: If you work hard, you will pass the exam. You did not pass. Therefore you did not work hard.
13. Without constructing the Truth Table prove that $(p \rightarrow q) \rightarrow q = p v q$?
14. Using normal forms, show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.
15. Show that $(x)\ (P(x) \vee Q(x)) \rightarrow (x)P(x) \vee (\exists x)Q(x)$
16. Show that $\neg (P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (\neg P \vee Q)$
    $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$
17. Prove that $(\exists x)\ (P(x) \wedge Q(x)) \rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$
18. Example: Prove or disprove the validity of the following arguments using the rules of inference. (i) All men are fallible (ii) All kings are men (iii) Therefore, all kings are fallible.
19. Test the Validity of the Following argument:
    "Lions are dangerous animals, there are lions, and therefore there are dangerous animals."

MULTIPLE CHOICE QUESTIONS

1: Which of the following propositions is tautology?
A.$(p \vee q) \rightarrow q$ B. $p \vee (q \rightarrow p)$ C.$p \vee (p \rightarrow q)$ D.Both (b) & (c)
Option: C

2: Which of the proposition is $p \wedge (\sim p \vee q)$ is
A.A tautology B.A contradiction C.Logically equivalent to $p \wedge q$ D.All of above
Option: C

3: Which of the following is/are tautology?
A.$a \vee b \rightarrow b \wedge c$ B.$a \wedge b \rightarrow b \vee c$ C.$a \vee b \rightarrow (b \rightarrow c)$ D.None of these
Option: B

4: Logical expression $(A \wedge B) \rightarrow (C' \wedge A) \rightarrow (A \equiv 1)$ is
A.ContradictionB.Valid C.Well-formed formula D.None of these
Option: D

5: Identify the valid conclusion from the premises $P \vee Q$, $Q \rightarrow R$, $P \rightarrow M$, $\neg M$
A.$P \wedge (R \vee R)$ B.$P \wedge (P \wedge R)$ C.$R \wedge (P \vee Q)$ D.$Q \wedge (P \vee R)$
Option: D

6: Let a, b, c, d be propositions. Assume that the equivalence $a \leftrightarrow (b \vee \neg b)$ and $b \leftrightarrow c$ hold. Then truth value of the formula $(a \wedge b) \rightarrow ((a \wedge c) \vee d)$ is always
A.True B.False C.Same as the truth value of a D.Same as the truth value of b
Option: A

7: Which of the following is a declarative statement?
A. It's right B. He says C.Two may not be an even integer D.I love you
Option: B

8: $P \rightarrow (Q \rightarrow R)$ is equivalent to
A. $(P \wedge Q) \rightarrow R$ B.$(P \vee Q) \rightarrow R$ C.$(P \vee Q) \rightarrow \neg R$ D.None of these
Option: A

9: Which of the following are tautologies?
A.$((P \vee Q) \wedge Q) \leftrightarrow Q$ B.$((P \vee Q) \wedge \neg P) \rightarrow Q$ C.$((P \vee Q) \wedge P) \rightarrow P$ D.Both (a) & (b)
Option: D

10: If F1, F2 and F3 are propositional formulae such that $F1 \wedge F2 \rightarrow F3$ and $F1 \wedge F2 \rightarrow F3$ are both tautologies, then which of the following is TRUE?
A.Both F1 and F2 are tautologies B.The conjuction $F1 \wedge F2$ is not satisfiable
C.Neither is tautologies D.None of these